CERN

U.S. Particle Accelerator School
Education in Beam Physics and Accelerator Technology

Collective effects in Beam Dynamics

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USPAS, one-week course, 19-24 January, 2015
http://uspas.fnal.gov/index.shtml

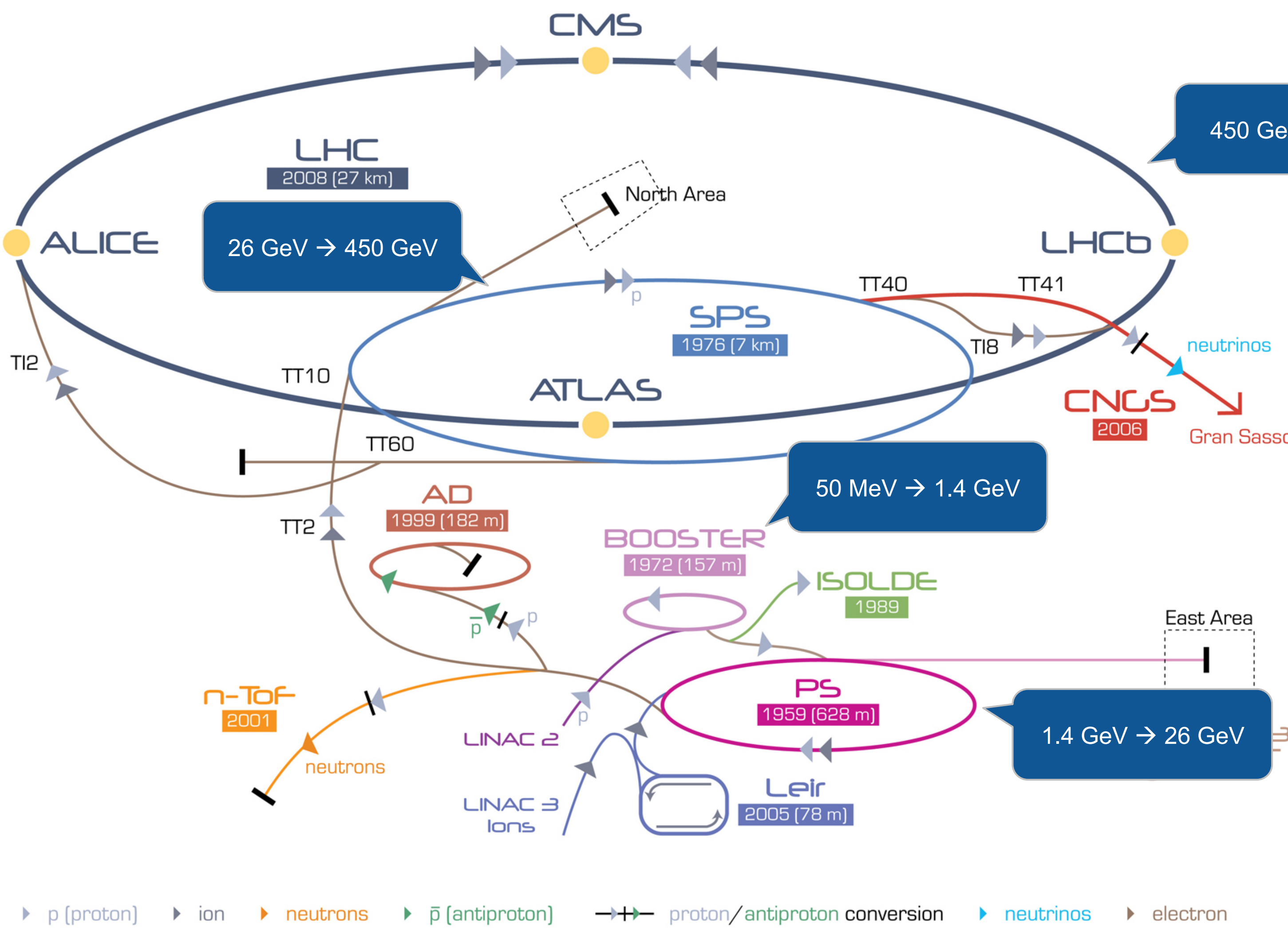
Outline

1. Introduction
   - Collective effects
   - Transverse single particle dynamics including systems of many non-interacting particles
   - Longitudinal single particle dynamics including systems of many non-interacting particles

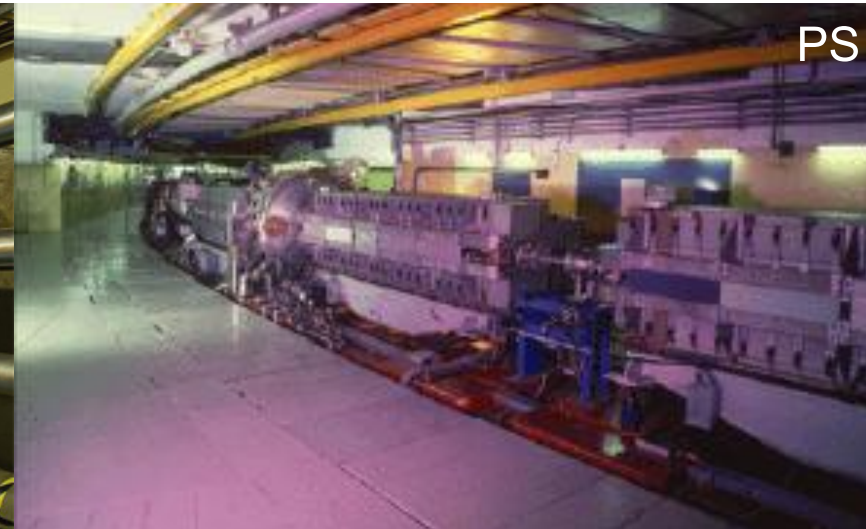
Introduction

- In this lecture we will treat the transverse motion of beam particles
- We focus on circular machines (in fact synchrotrons)
- In the course of this lecture series we will encounter several examples of collective effects observed in the CERN accelerator complex
  - Proton Synchrotron Booster (PSB)
  - Proton Synchrotron (PS)
  - Super Proton Synchrotron (SPS)
  - Large Hadron Collider (LHC)

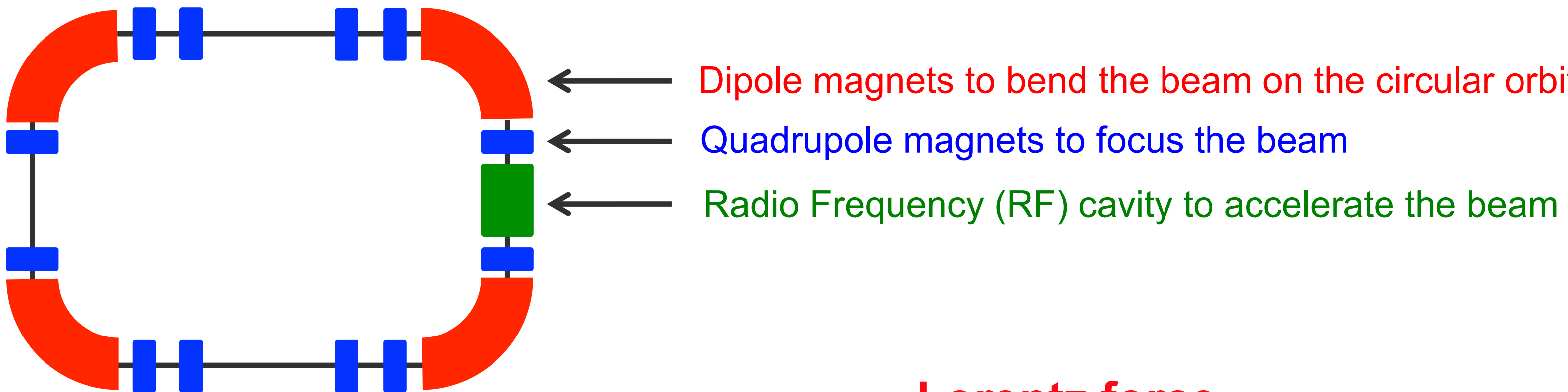
CERN accelerator complex



The CERN machines



What is a synchrotron?



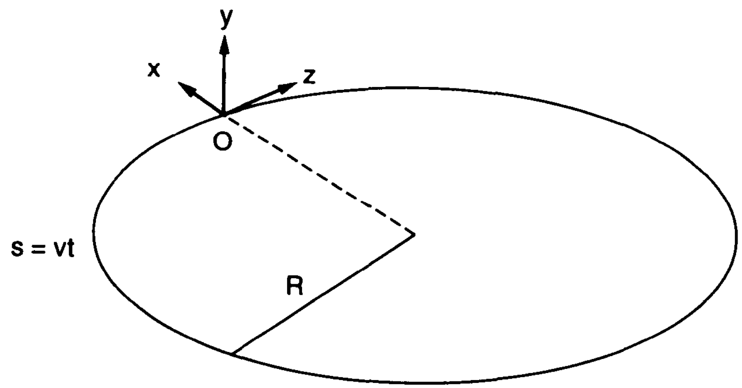
Lorentz force

- Motion of single particle is described by

$$\boxed{\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})}$$

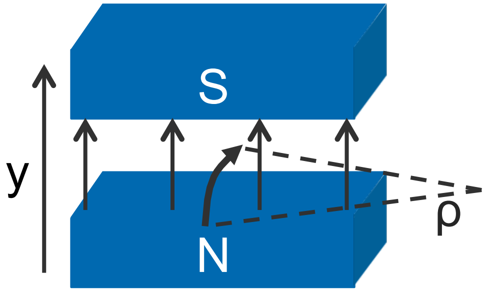
- Main characteristics of synchrotrons
  - Use electric fields to accelerate and magnetic fields to guide particles
  - **Design orbit is fixed** at a given radius **independent of the beam energy** (magnetic field is increased proportional to momentum)
  - Beam **is accelerated during many revolutions** passing through the same RF cavity
  - Accelerating RF is **synchronized** with particle revolution frequency → “Synchrotron”

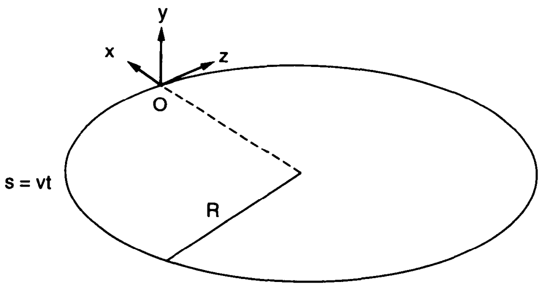
Coordinate system



- We use a co-moving coordinate system to describe the particle motion around the reference orbit
  - The origin **O** is moving along with “synchronous particle”, i.e. a reference particle that has the design momentum and follows the design orbit
  - Mean radius R is defined through machine circumference $C = 2\pi R$
  - Transverse coordinates **x** and **y** relative to reference particle (where x,y << R)
  - Longitudinal coordinate **z** relative to reference particle
  - Position along accelerator is described by independent variable **s = vt**

Dipole magnets – beam guidance





dipole magnets: uniform magnetic field in y direction

- In a uniform magnetic field *B,* a particle with charge *e*, velocity *v*, rest mass *m* and Lorentz factor γ follows a circular trajectory with bending radius ρ

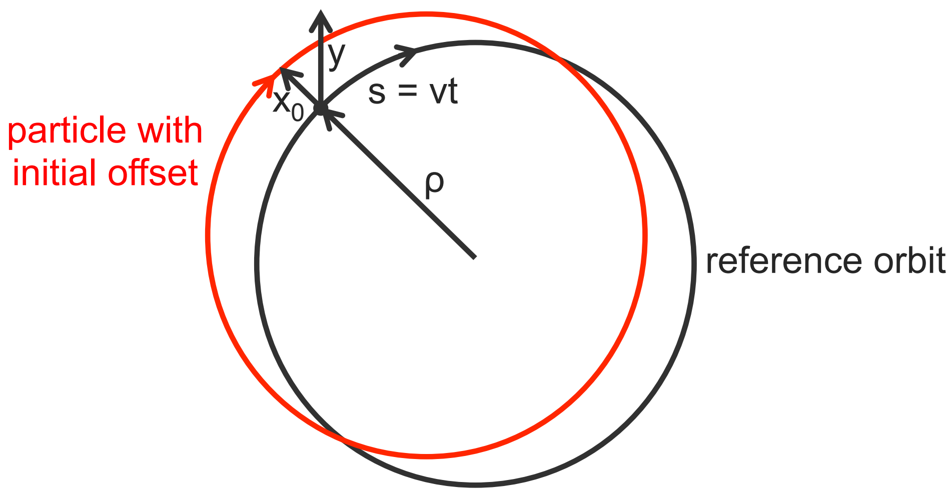
$$\underbrace{evB}_{\text{Lorentz force}} = \underbrace{\frac{\gamma m v^2}{\rho}}_{\text{Centrifugal force}} \longrightarrow \boxed{B\rho = \frac{p}{e}}$$

magnetic rigidity

- The magnetic field of dipole magnets in a synchrotron defines
  - the reference trajectory (orbit) around the machine
  - the reference momentum (through the magnetic rigidity)

Dipole magnets – weak focusing

- Consider a particle with initial offset from reference orbit in a **uniform magnetic field** in y direction

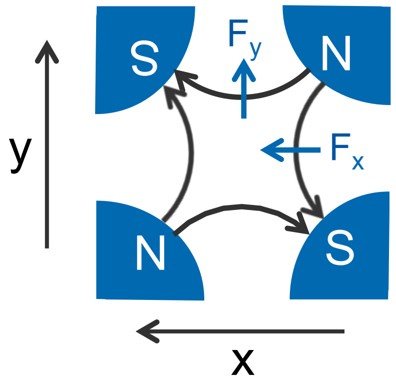


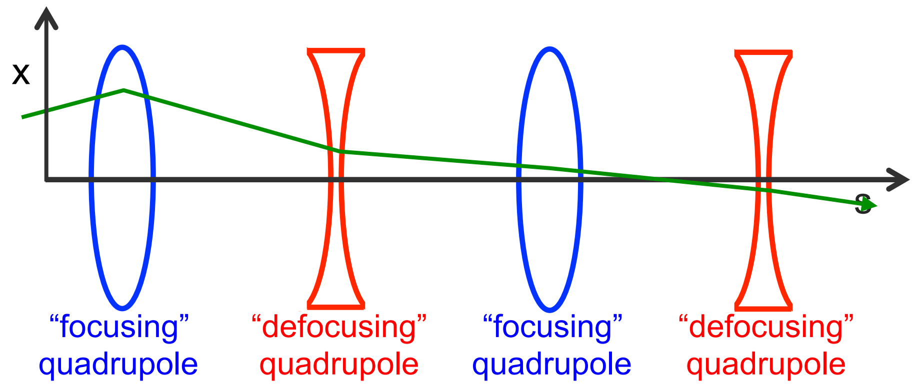
- The particle performs a harmonic oscillation with frequency ω

$$x = x_0 \cos(\omega t + \phi) \qquad \omega = \frac{v}{\rho}$$

- **This is the weak focusing** in **horizontal** plane: $$x'' \equiv \frac{d^2 x}{ds^2} = \frac{d^2 x}{v^2 dt^2} = -\frac{1}{\rho^2} x$$

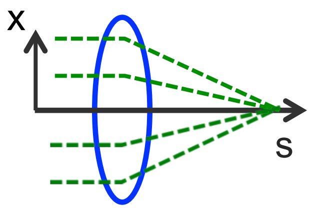
Quadrupole magnets – strong focusing





- Magnetic field proportional to offset results in linear restoring force

$$B_y = -gx \qquad F_x = -evgx$$
$$B_x = -gy \quad \longrightarrow \quad F_y = +evgy$$



- Force is focusing in one plane while defocusing in the other → need to alternate between focusing and defocusing quadrupoles ("alternating gradient lattice") to achieve overall focusing in a particle accelerator or transfer line
- In accelerator design we use the normalized quadrupole strength $K = \dfrac{g}{B\rho}$

Equations of motion

- Consider linear fields (dipoles + quadrupoles) and **on-momentum particles**

quadrupoles　　dipoles ("weak focusing")

$$\left.\begin{aligned} x'' - \left(K(s) - \frac{1}{\rho(s)^2}\right) x = 0 \\ y'' + K(s)\, y = 0 \end{aligned}\right\}$$

can be written as:

Hill's equations

$$\boxed{\begin{aligned} x'' + K_x(s)\, x = 0 \\ y'' + K_y(s)\, y = 0 \end{aligned}}$$

- Linear equations with *s*-dependent coefficients
  - equivalent to harmonic oscillator with *s*-dependent dependent frequency
  - in a ring (or transport line with symmetries), the focusing terms are periodic:

$$K_x(s) = K_x(C + s) \qquad K_y(s) = K_y(C + s)$$

- Not straightforward to derive analytical solutions for whole accelerator …

Element-wise solution of Hill's equations

- Consider a part of the accelerator where focusing term is constant: $K=K_0$

$$u'' + K_0 u = 0$$ … u stands for x or y

- This is the equation of a harmonic oscillator with the element-wise solutions

$$u(s) = C(s)u(0) + S(s)u'(0)$$
$$u'(s) = C'(s)u(0) + S'(s)u'(0)$$

… $u(0)$ and $u'(0)$ are the initial conditions

where
$$\begin{cases} C(s) = \cos\left(\sqrt{K_0}s\right), & S(s) = \frac{1}{\sqrt{K_0}}\sin\left(\sqrt{K_0}s\right) & \text{for } K_0 > 0 \text{ (focusing)} \\ C(s) = 1, & S(s) = s & \text{for } K_0 = 0 \text{ (drift)} \\ C(s) = \cosh\left(\sqrt{|K_0|}s\right), & S(s) = \frac{1}{\sqrt{|K_0|}}\sinh\left(\sqrt{|K_0|}s\right) & \text{for } K_0 < 0 \text{ (defocusing)} \end{cases}$$

- In general the solution can be written in **matrix form**

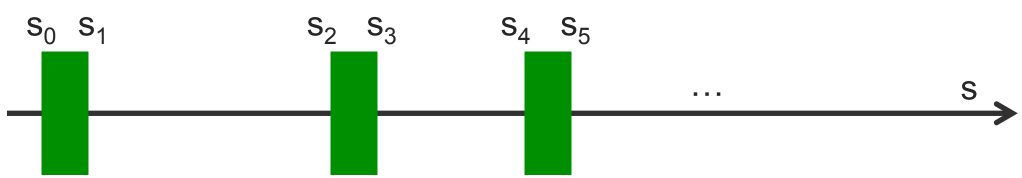
$$\begin{pmatrix} u(s) \\ u'(s) \end{pmatrix} = \begin{pmatrix} C(s) & S(s) \\ C'(s) & S'(s) \end{pmatrix} \begin{pmatrix} u(0) \\ u'(0) \end{pmatrix}$$

Transfer matrix formalism

- The general transfer matrix from location s_0 to s is written as

$$\begin{pmatrix} u \\ u' \end{pmatrix}_s = \mathcal{M}(s|s_0) \begin{pmatrix} u \\ u' \end{pmatrix}_{s_0} = \begin{pmatrix} C(s|s_0) & S(s|s_0) \\ C'(s|s_0) & S'(s|s_0) \end{pmatrix} \begin{pmatrix} u \\ u' \end{pmatrix}_{s_0}$$

- The transport through an accelerator or transfer line can be described by a series of matrix multiplications



$$\mathcal{M}(s_n|s_0) = \underbrace{\mathcal{M}(s_n|s_{n-1}) \ldots \underbrace{\mathcal{M}(s_2|s_1) \cdot \underbrace{\mathcal{M}(s_1|s_0)}_{\text{from } s_0 \text{ to } s_1}}_{\text{from } s_0 \text{ to } s_2}}_{\text{from } s_0 \text{ to } s_n}$$

General solution of Hill's equation

- The **general solution of Hill's equations** ("betatron motion") can be written as

$$\boxed{\begin{aligned} u(s) &= \sqrt{2J_u\beta_u(s)}\cos\left(\psi_u(s)+\psi_u(s_0)\right) \\ u'(s) &= -\sqrt{\frac{2J_u}{\beta_u(s)}}\left[\alpha_u(s)\cos\left(\psi_u(s)+\psi_u(s_0)\right)+\sin\left(\psi_u(s)+\psi_u(s_0)\right)\right] \end{aligned}}$$

$$\beta_u(s), \quad \alpha_u(s) = -\frac{\beta_u'(s)}{2}, \quad \gamma_u(s) = \frac{1+\alpha_u(s)^2}{\beta_u(s)} \qquad \psi_u(s) = \int \frac{ds}{\beta_u(s)}$$

"Twiss" parameters at s

Betatron phase

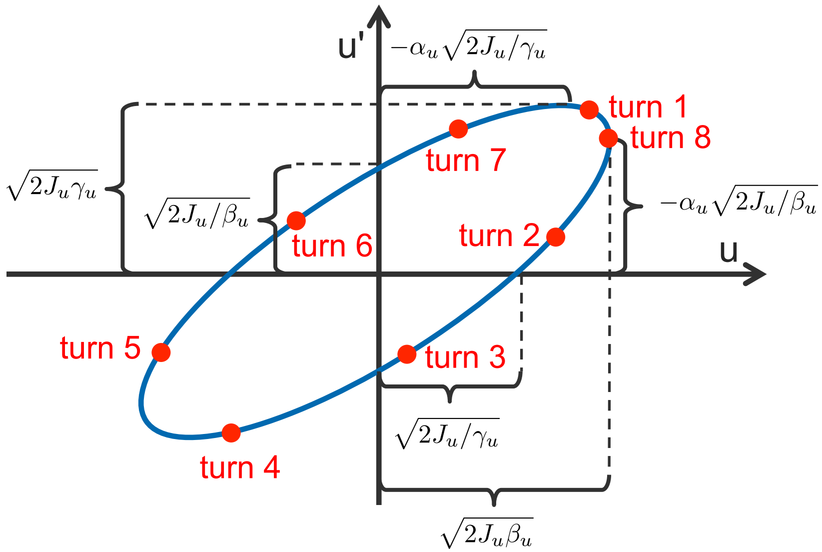
- The beta function is defined by the **envelope equation** (follows from Hill's equation)

$$2\beta_u\beta_u'' - \beta_u'^2 + 4\beta_u^2 K_u = 0$$

- The "action" ***J_u*** **is a constant of motion** (i.e. independent of *s*)

$$2J_u = \gamma_u u^2 + 2\alpha_u u u' + \beta_u u'^2$$

Phase space ellipse



- The **phase space coordinates** (*u, u'*) of a single particle at a given location *s* in the machine **lie on the phase space ellipse** when plotted for several turns.
- The **values of the Twiss parameters** and therefore the orientation of the phase space ellipse **depend on the *s* location** in the machine. The phase space area enclosed by the ellipse is invariant and equal to $2J_u\pi$.
- The **Twiss parameters are periodic with the machine circumference**. Their values are derived from the transfer matrix and they are uniquely defined at any point in the machine.

General transfer matrix

- From the general solutions for u and u' we can write

$$\cos\left(\psi_u(s)+\psi_u(s_0)\right)=\frac{u(s)}{\sqrt{2J_u\beta_u(s)}} \qquad \sin\left(\psi_u(s)+\psi_u(s_0)\right)=\sqrt{\frac{\beta_u(s)}{2J_u}}u'(s)+\frac{\alpha_u(s)}{\sqrt{2J_u\beta_u(s)}}u(s)$$

- The **general transfer matrix** from location s_0=0 to s is obtained as

$$\begin{pmatrix} u(s) \\ u'(s) \end{pmatrix} = \mathcal{M}_u(s|s_0)\begin{pmatrix} u(s_0) \\ u'(s_0) \end{pmatrix}$$

$$\boxed{\mathcal{M}_u(s|s_0)=\begin{pmatrix} \sqrt{\frac{\beta_u(s)}{\beta_u(s_0)}}(\cos\Delta\psi_u+\alpha_u(s_0)\sin\Delta\psi_u) & \sqrt{\beta_u(s_0)\beta_u(s)}\sin\Delta\psi_u \\ \frac{\alpha_u(s_0)-\alpha_u(s)}{\sqrt{\beta_u(s_0)\beta_u(s)}}\cos\Delta\psi_u-\frac{1+\alpha_u(s_0)\alpha_u(s)}{\sqrt{\beta_u(s_0)\beta_u(s)}}\sin\Delta\psi_u & \sqrt{\frac{\beta_u(s_0)}{\beta_u(s)}}(\cos\Delta\psi_u-\alpha_u(s)\sin\Delta\psi_u) \end{pmatrix}}$$

$$\Delta\psi_u=\int_0^s\frac{ds}{\beta_u(s)}$$ **… betatron phase advance**

- Note: for a given part of the accelerator, this general transfer matrix based on beta functions is equivalent to the transfer matrix written in terms of K(s) obtained earlier from the multiplication of element wise solutions

Periodic transfer matrix

- Now we consider a periodic structure, in particular the **transfer matrix for a full machine circumference *C***
  - the optics functions must be *periodic* and are therefore *uniquely defined*!

$$\beta_u(0) = \beta_u(C) = \beta_u \qquad \alpha_u(0) = \alpha_u(C) = \alpha_u$$

  - The phase advance over one turn is usually expressed as the **betatron tune Q_u**, which corresponds to the **number of betatron oscillations per turn**

$$\phi_u = \int_0^C \frac{ds}{\beta_u(s)} \longrightarrow \boxed{Q_u \equiv \frac{1}{2\pi}\int_0^C \frac{ds}{\beta_u(s)}}$$

- The one turn transfer matrix is obtained as

$$\boxed{\mathcal{M}_u(C|0) = \begin{pmatrix} \cos\phi_u + \alpha_u \sin\phi_u & \beta_u \sin\phi_u \\ -\frac{1+\alpha_u^2}{\beta_u}\sin\phi_u & \cos\phi_u - \alpha_u \sin\phi_u \end{pmatrix}}$$

Smooth approximation

- An estimation of the average beta function from the betatron tune (or conversely an estimation of the tune from the average beta function) for an accelerator with radius *R* can be obtained by:

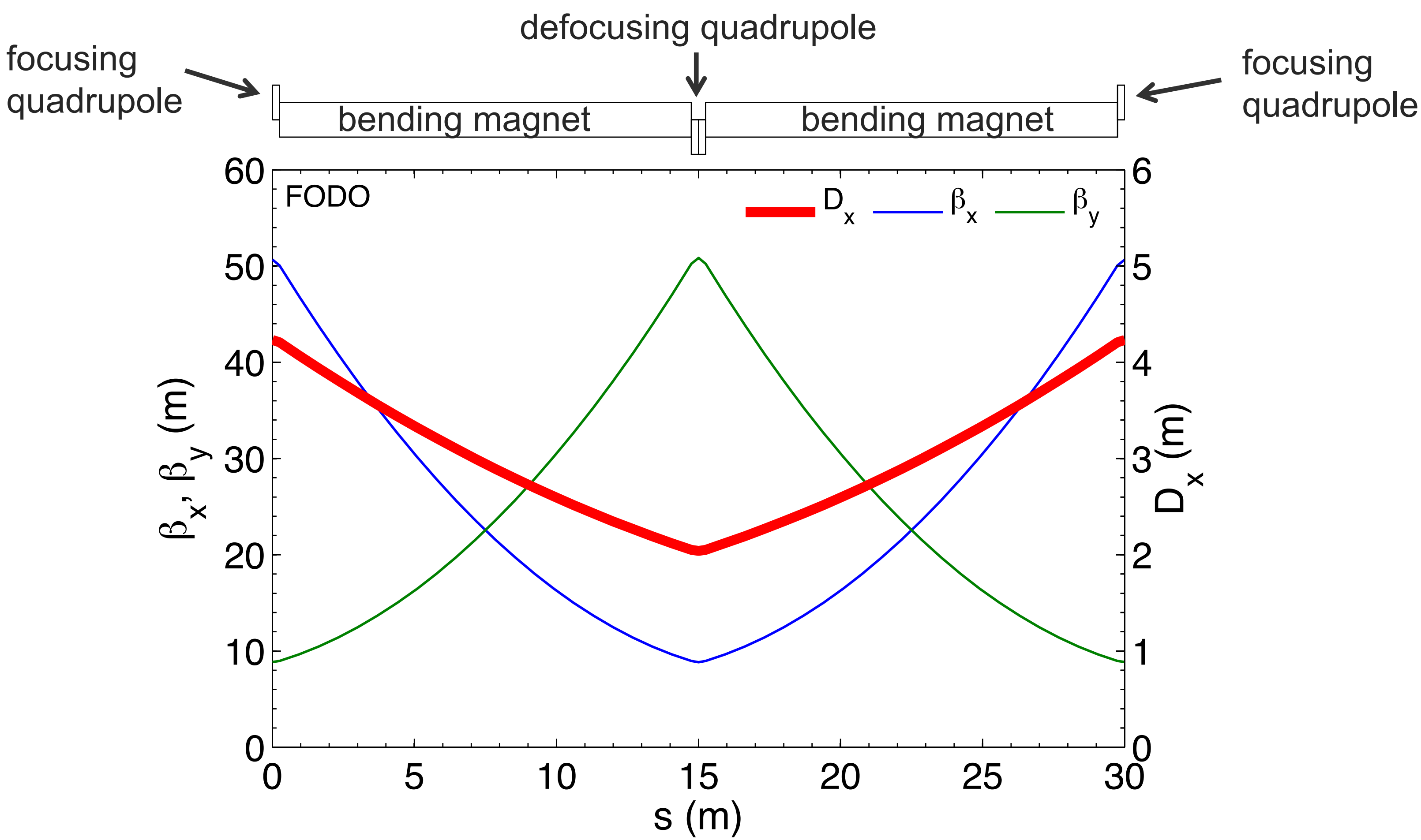
$$Q_u = \frac{1}{2\pi}\int_0^C \frac{ds}{\beta_u(s)} \approx \frac{2\pi R}{2\pi}\frac{1}{\langle\beta_u\rangle} = \frac{R}{\langle\beta_u\rangle}$$

$$\boxed{Q_u \approx \frac{R}{\langle\beta_u\rangle}}$$

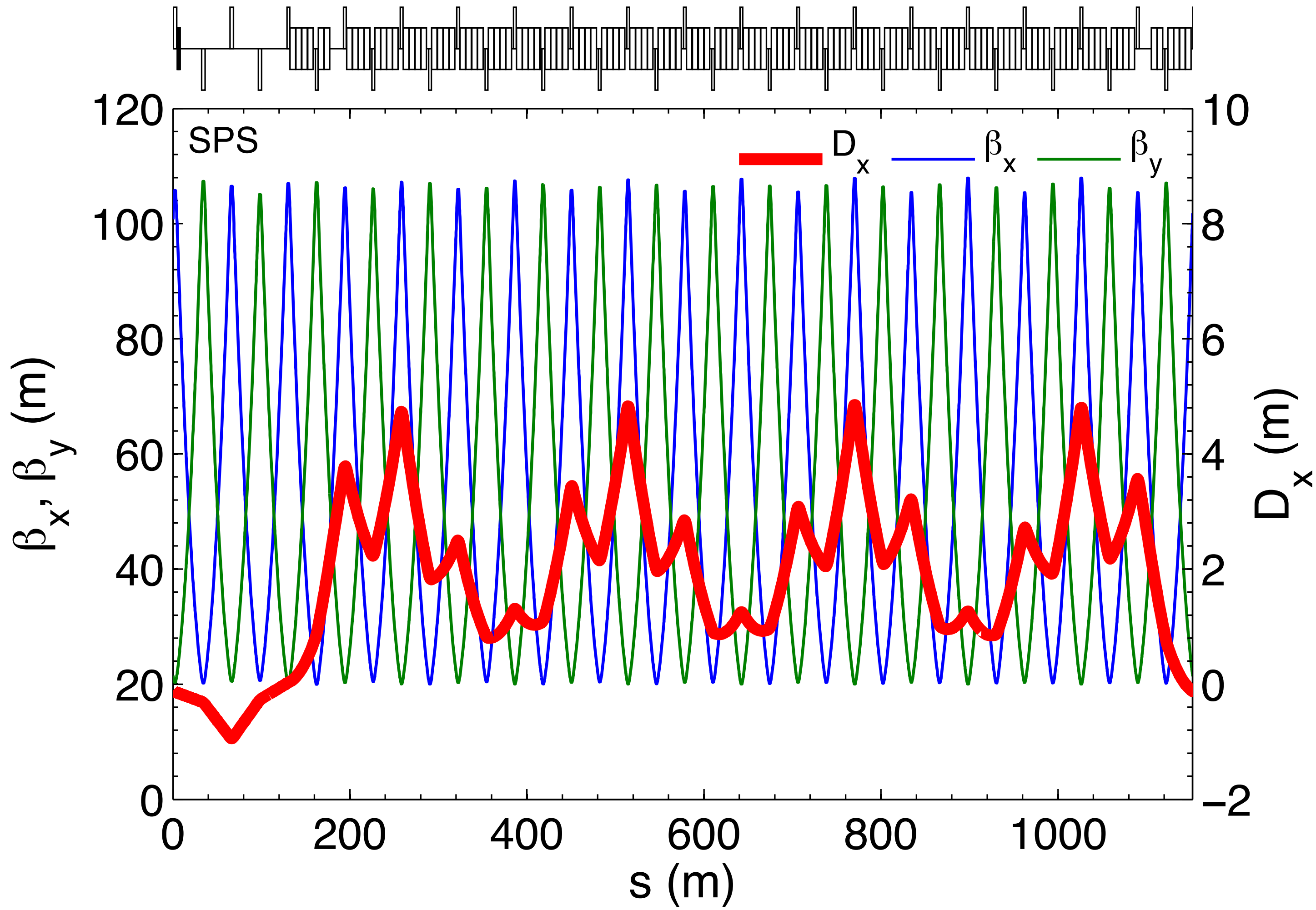
- Corresponds to a **uniform focusing** channel and is often used for quick calculations and in particular for the theoretical treatment of transverse beam instabilities. Since the betatron tune indicates the number of transverse oscillations per turn, the Hill's equation in smooth approximation can also be written as

$$u'' + \left(\frac{Q_u}{R}\right)^2 u = 0$$

Example: FODO cell



Example: lattice of the SPS



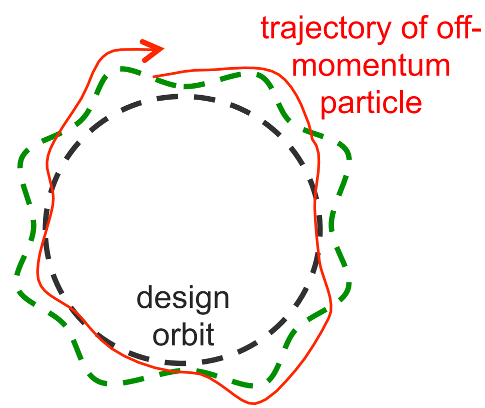
Off-momentum particles – dispersion

- Consider a particle having a momentum error *Δp* w.r.t. the reference particle
- In a dipole magnet, off momentum particles have a different bending angle and thus receive a different deflection compared to the reference particle. Off-momentum particles therefore follow a different closed orbit along the machine.
- The equation of motion in the horizontal plane becomes $\boxed{x'' + K_x(s)\,x = \dfrac{1}{\rho(s)}\dfrac{\Delta p}{p}}$
- Solution is sum of homogeneous and inhomogeneous part

$$x(s) = \underbrace{\sqrt{2J_x\beta_x(s)}\cos\left(\psi_x(s) + \psi_x(s_0)\right)}_{\text{solution of } x'' + K_x(s)\,x = 0} + \underbrace{D_x(s)\frac{\Delta p}{p}}_{\text{particular solution with “dispersion” } D_x}$$

- Inserting *x(s)* into equation of motion yields

$$\boxed{D_x''(s) + K_x(s)D_x(s) = \frac{1}{\rho(s)}}$$



$D_x(s)$*Δp/p defines the closed orbit for off-momentum particles

Momentum compaction

- The closed orbit for an individual particle depends on its momentum offset with respect to the reference particle

$$\boxed{\delta \equiv \Delta p/p}$$

- To the lowest order in δ, the change of the circumference is given by

$$\Delta C = \oint \frac{x}{\rho(s)} ds = \left[\oint \frac{D_x(s)}{\rho(s)} ds \right] \delta$$

- The **momentum compaction factor** α_0 relates the relative change of the circumference to the relative momentum change

$$\alpha_0 \equiv \frac{\Delta C/C}{\Delta p/p_0} = \frac{1}{C} \oint \frac{D_x(s)}{\rho(s)} ds$$

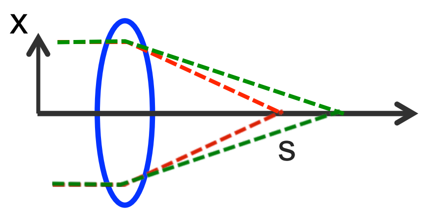
in a FODO lattice

$$\alpha_0 \approx \frac{1}{{Q_x}^2}$$

- The momentum compaction factor plays a central role for the longitudinal beam dynamics (see Kevin's lecture)

Off-momentum particles – chromaticity

- Chromaticity is the dependence of the betatron tune on the relative momentum offset of a particle



change of focusing strength for particles with different momenta

relative momentum offset

$$\boxed{\delta \equiv \Delta p/p}$$

chromaticity

$$\boxed{\xi_u \equiv \frac{dQ_u}{d\delta}}$$

$$K = \frac{g}{B\rho} = \frac{eg}{p} \longrightarrow \frac{\Delta K}{K} = -\frac{\Delta p}{p}$$

$$\longrightarrow \Delta K = -\delta K$$

focusing error is prop. to relative momentum offset

$$\Delta Q_u = \frac{1}{4\pi}\oint \beta_u(s)\Delta K_u(s)ds \approx$$

$$\approx \left(-\frac{1}{4\pi}\oint \beta_u(s)K_u(s)ds\right)\delta$$

$$\longrightarrow \xi_u = -\frac{1}{4\pi}\oint \beta_u(s)K_u(s)ds$$

natural chromaticity is always < 0

- Chromaticity plays a fundamental role for transverse instabilities
  - In many cases chromaticity needs to be adjusted → this can be done by sextupole magnets installed in locations with non-zero dispersion

Betatron detuning with amplitude

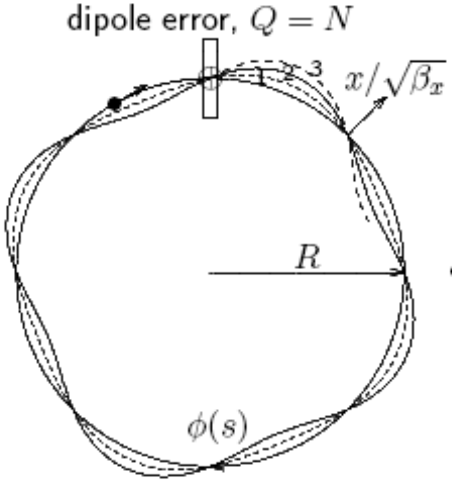
- Up to now we considered only linear magnetic fields (dipoles, quadrupoles) in the equations of motion
- In a real accelerator there are also **non-linear fields** (from sextupoles for chromaticity correction, from magnetic field imperfections, …)
  - In the presence of non-linear magnetic fields, the betatron tune Q_u depends on the betatron amplitude (action) J_u
  - To the first order in J_u the detuning with amplitude can be written as

$$\begin{pmatrix} \Delta Q_x \\ \Delta Q_y \end{pmatrix} = \underbrace{\begin{pmatrix} a_{xx} & a_{xy} \\ a_{yx} & a_{yy} \end{pmatrix}}_{\text{“anharmonicities”}} \begin{pmatrix} J_x \\ J_y \end{pmatrix}$$

  - **Octupole magnets** induce first order detuning with amplitude in leading order and can thus be used to adjust the anharmonicities
- In some cases detuning with amplitude is generated on purpose in order to fight instabilities (e.g. in the LHC) …

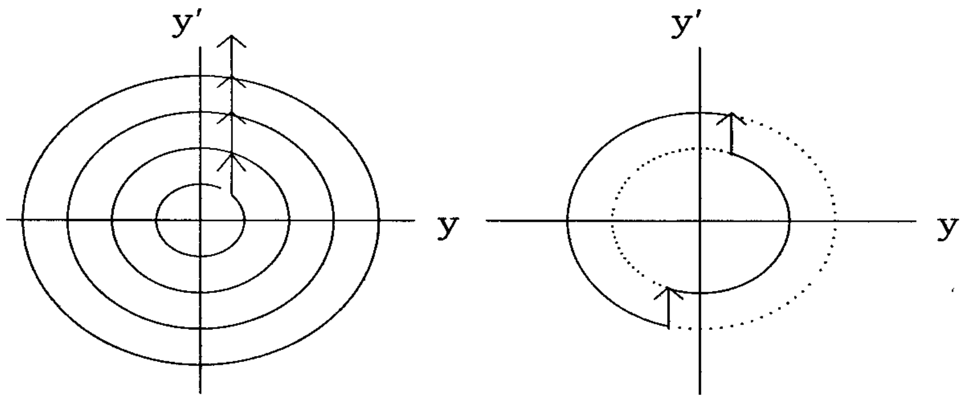
Resonances

- In the presence of optical machine imperfections the values of the betatron tunes should not be on or close to a rational fraction
  - Dipole errors deflect a particle each turn in phase if tune is an integer N



closed orbit instability

Effect of dipole error in phase space

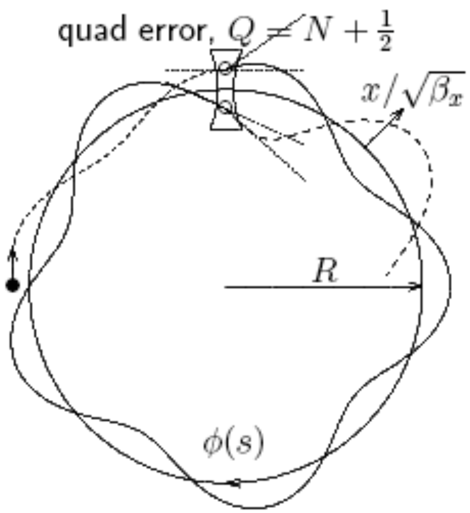


Q = N
dipole kicks add up

Q = N/2
cancellation
after two turns

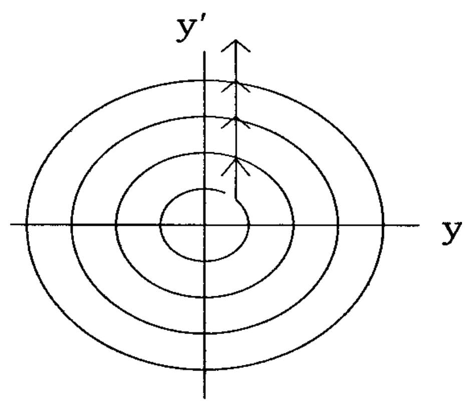
Resonances

- In the presence of optical machine imperfections the values of the betatron tunes should not be on or close to a rational fraction
  - Dipole errors deflect a particle each turn in phase if tune is an integer N
  - Quadrupole errors are in phase if tune is an integer N or a half integer N+1/2

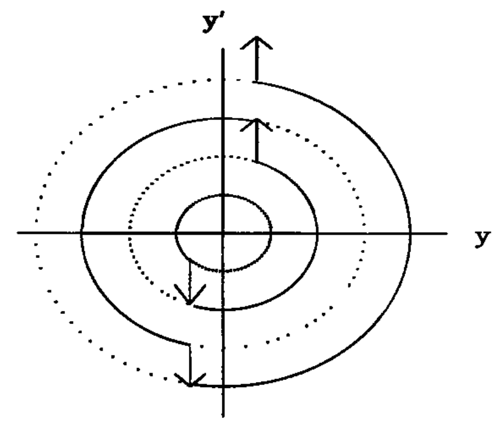


beam size grows each turn

Effect of quadrupole error in phase space



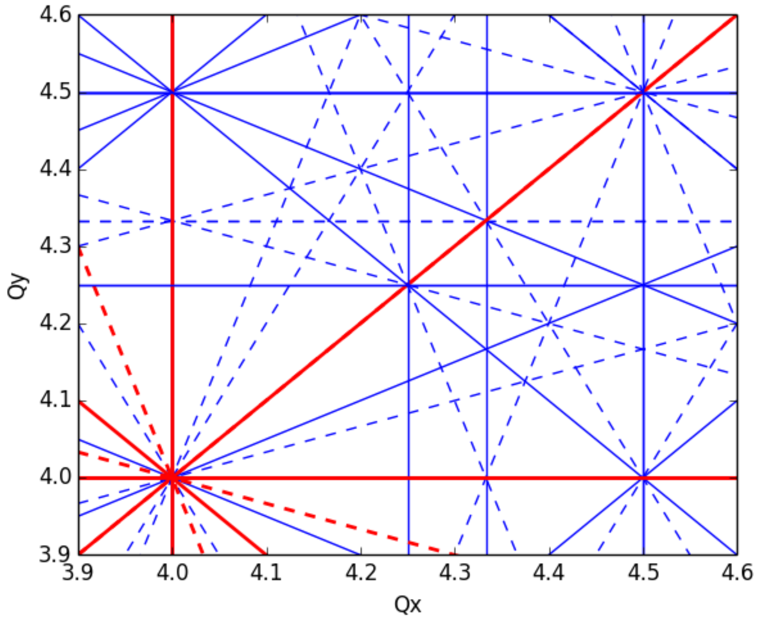
Q = N
quadrupole
kicks add up



Q = N/2
quadrupole
kicks add up

Resonances

- In the presence of optical machine imperfections the values of the betatron tunes should not be on or close to a rational fraction
  - Dipole errors deflect a particle each turn in phase if tune is an integer N
  - Quadrupole errors are in phase if tune is an integer N or a half integer N+1/2
  - The 2 dimensional resonance condition is $\boxed{kQ_x + lQ_y = m \text{ for } k, l, m \text{ integers}}$



$|k| + |l| \longrightarrow$ order of the resonance

The tune diagram shows the resonance lines where the betatron motion can be unstable (here up to 4th order)

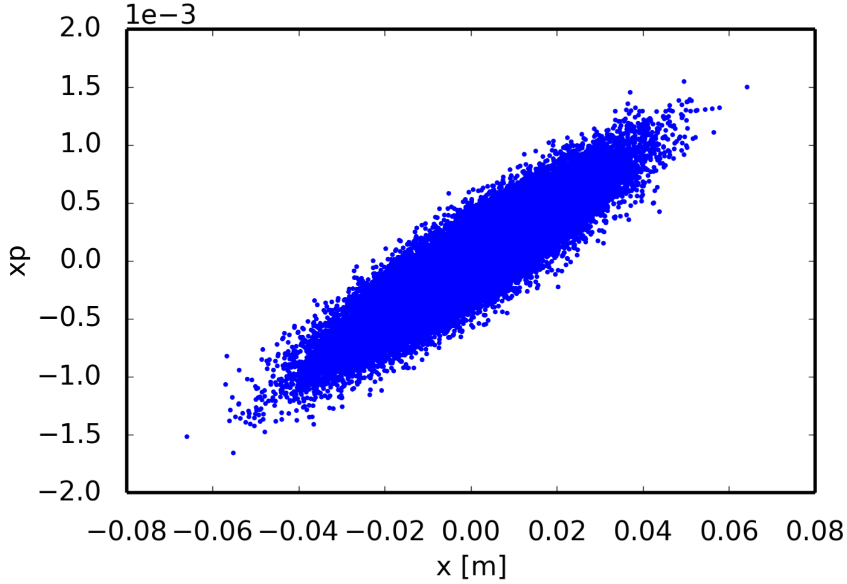
Usually the strength of the resonance decreases as the resonance order increases

The "working point" corresponds to the tunes of the machine (as defined by the focusing)

Particle ensemble

- Up to now we were looking at individual particles …
- Let's have a look at a beam consisting of N particles which are described by a particle distribution function

$$\int \psi(u,u')\,du\,du' = N$$



statistical moments of the distribution

$$\langle u\rangle = \frac{1}{N}\int u\psi(u,u')\,du\,du'$$

$$\langle u'\rangle = \frac{1}{N}\int u'\psi(u,u')\,du\,du'$$

$$\sigma_u^2 = \frac{1}{N}\int (u-\langle u\rangle)^2\,\psi(u,u')\,du\,du'$$

$$\sigma_{u'}^2 = \frac{1}{N}\int (u'-\langle u'\rangle)^2\,\psi(u,u')\,du\,du'$$

$$\sigma_{uu'} = \frac{1}{N}\int (u-\langle u\rangle)\,(u'-\langle u'\rangle)\,\psi(u,u')\,du\,du'$$

Transverse emittance

- The **beam size in the u-u' phase space** is usually quantified by the **rms statistical emittance** (also called geometrical emittance)

$$\boxed{\varepsilon_u = \sqrt{\sigma_u^2 \sigma_{u'}^2 - \sigma_{uu'}^2}}$$

  - The transverse momentum in the accelerator is given by $p_u = m\beta c\gamma u'$
  - The Liouville theorem states that volumes in the canonical phase space **u – p_u** are invariant if their evolution is governed by a Hamiltonian (like beam transport through an accelerator) → therefore the geometrical emittance (defined in the **u-u'** phase space) shrinks during acceleration ("adiabatic damping")

- We define the **normalized emittance**, which is independent of beam energy

$$\boxed{\varepsilon_u^n = \beta\gamma\varepsilon_u}$$

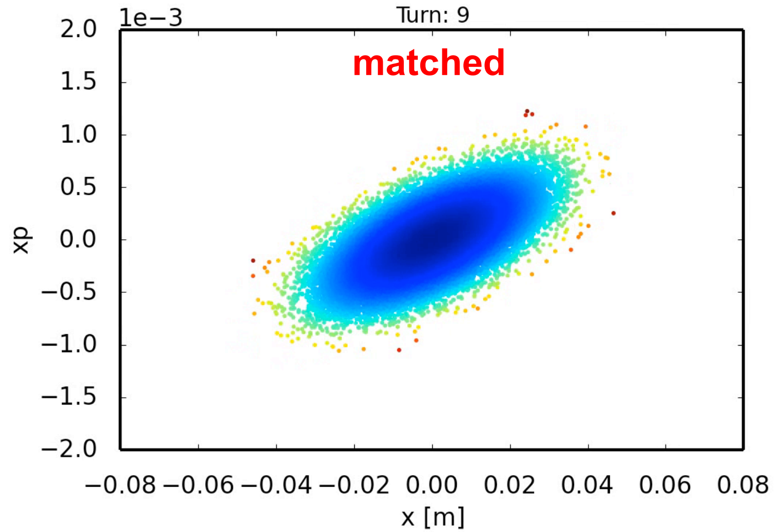
- However there are effects that can lead to emittance blow-up, such as scattering effects, filamentation due to non-linearities, wake fields, space charge effects, …

Transverse phase space matching

- We can calculate the statistical TWISS parameters of a beam like

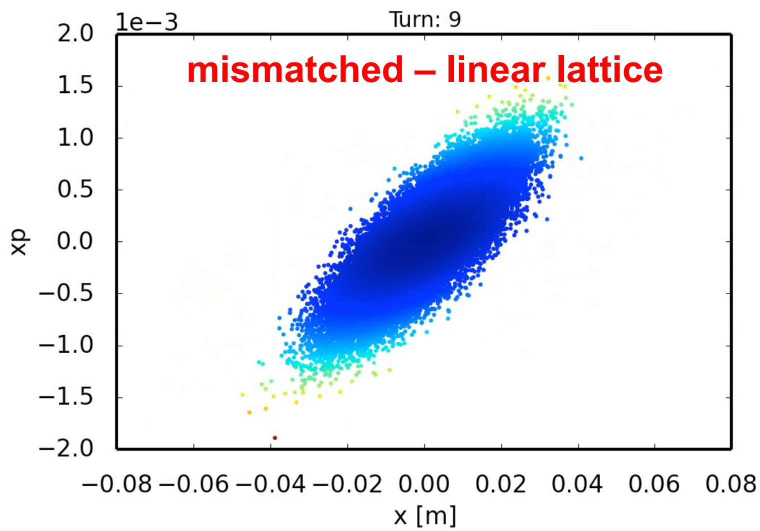
$$\alpha_u^{\text{beam}} = -\frac{\sigma_{uu'}}{\varepsilon_u} \qquad \beta_u^{\text{beam}} = \frac{\sigma_u^2}{\varepsilon_u} \qquad \gamma_u^{\text{beam}} = \frac{\sigma_{u'}^2}{\varepsilon_u}$$

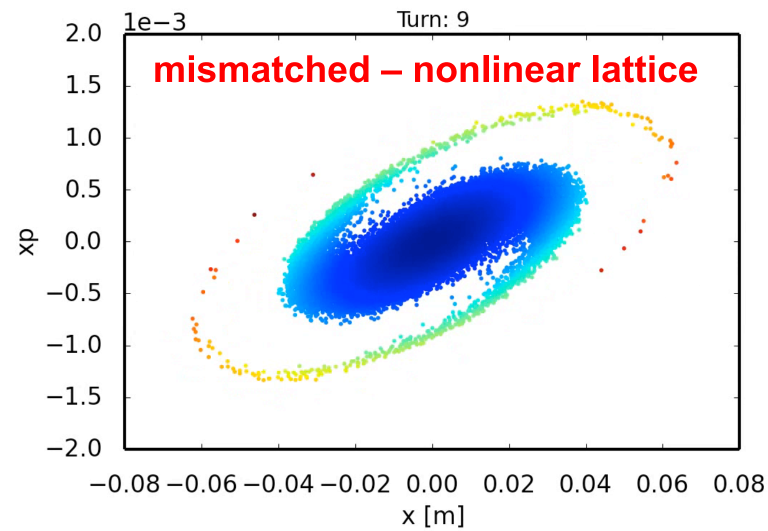
- A beam is matched to the optics at the injection point of a machine means that the TWISS parameters of the beam are the same as the one of the machine
  - If the beam is matched, the shape of the particle distribution remains stationary from turn to turn



Filamentation

- Injecting a mismatched beam results in quadrupole oscillations, i.e. the shape of the particle distribution will change from turn to turn
  - In a linear lattice (with zero chromaticity), all particles have the same tune → the bunch will perform quadrupole oscillations but the emittance is preserved
  - In a non-linear lattice, the tune of a particle depends on its betatron amplitude → the result is a dilution of the phase space area covered by the particle distribution, i.e. the emittance grows and the beam quality is degraded





Summary

- The linear transverse particle motion is described by Hill's equations
  - The beam transport around the accelerator can be represented by "transfer" matrices
  - The number of oscillations around the closed orbit is called betatron tune *Q*
  - The particle motion is described by a pseudo-harmonic oscillation with varying amplitude (beta-function) and phase (betatron phase advance)
  - In general the beam envelope (beam size) is varying around the machine
  - The smooth approximation assumes a uniform focusing force around the accelerator
- Off momentum particles follow a different closed orbit (dispersion) and can have a different tune in case the chromaticity is not corrected (using sextupoles)
- Non-linear elements (e.g. octupoles) introduce betatron detuning with amplitude
- To avoid beam envelope oscillations and emittance growth (in case of non-linear magnetic elements), the beam distribution has to be matched to the machine optics at the injection point