Introduction

Single particle dynamics in the longitudinal plane

Outline

1. Introduction
    - Collective effects
    - Transverse single particle dynamics including systems of many non-interacting particles
    - Longitudinal single particle dynamics including systems of many non-interacting particles

Longitudinal dynamics

A particle's longitudinal coordinates are given with respect to the two reference coordinates.

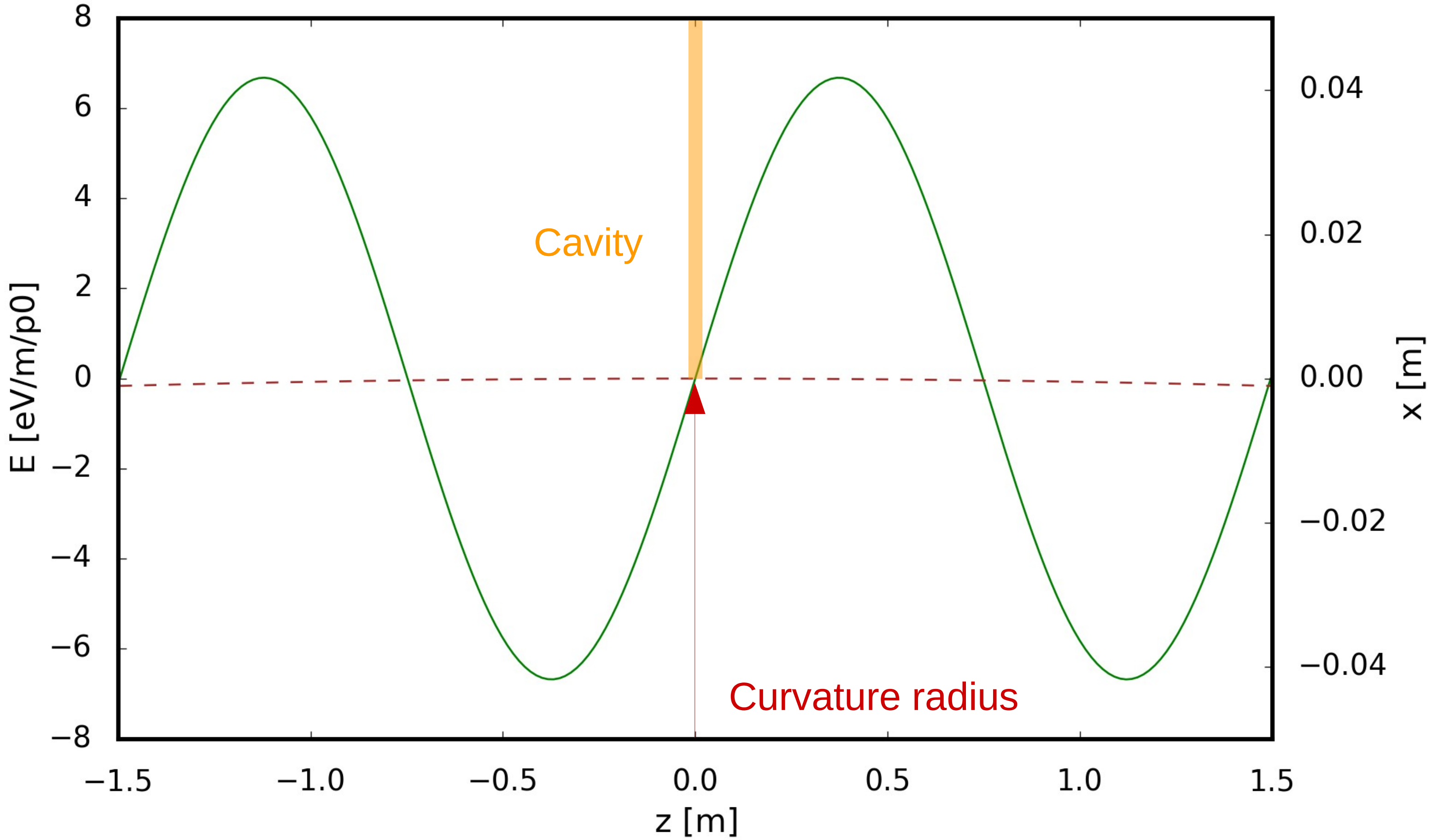
- The reference momentum p_0 is defined by the magnetic rigidity

$$B\rho = \frac{p_0}{e}$$

- The reference position is defined by the global fundamental RF clock via the time of zero crossing of the fundamental RF voltage

Longitudinal dynamics

- We take a snapshot of the beam at the RF cavity
- $t = 0$ is the time of zero crossing of the fundamental RF voltage



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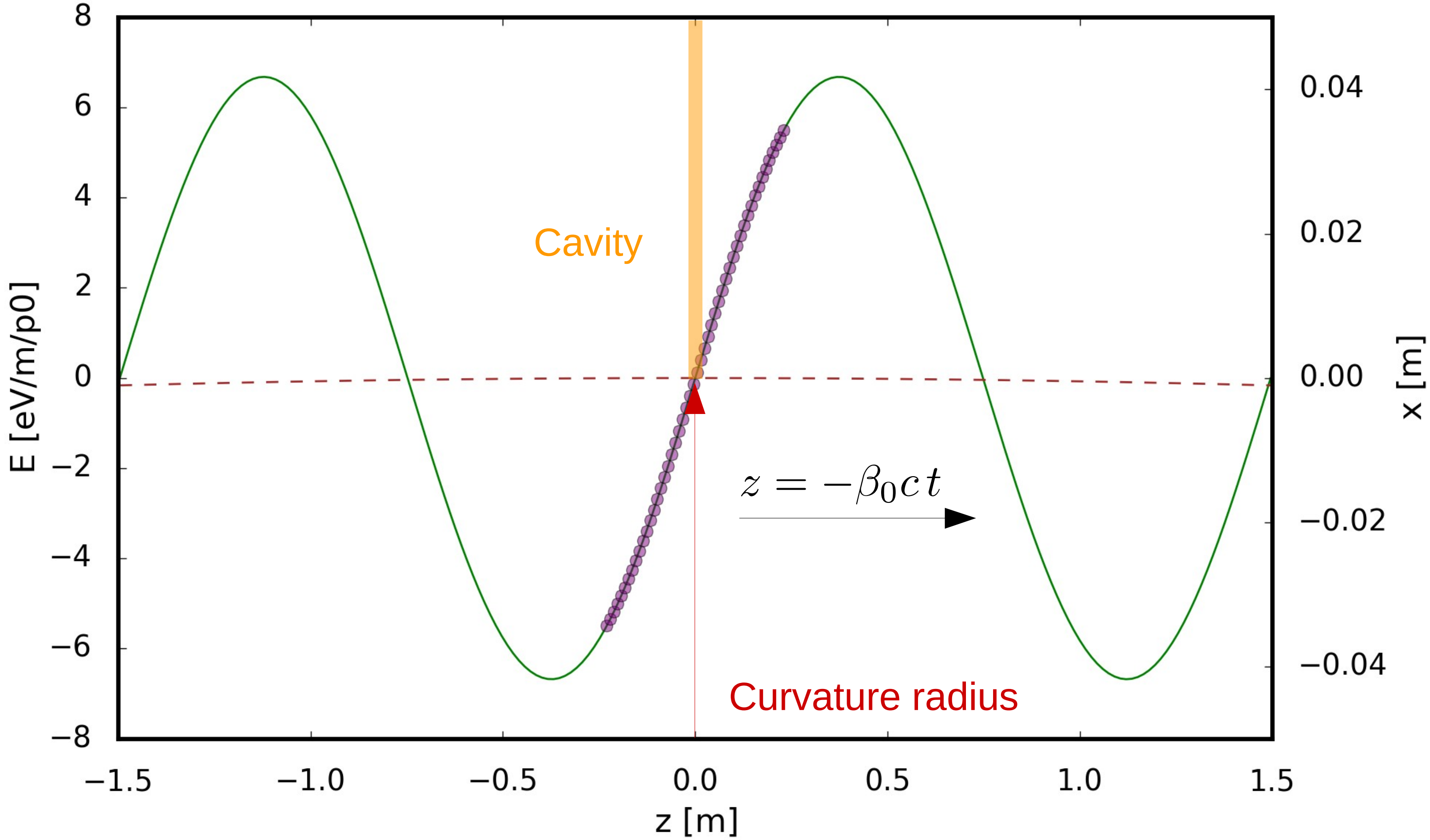

Longitudinal dynamics - drift

- The bending fields B and the reference momentum p_0 fix the design orbit C_0
- Momentum deviations result in a change of the design orbit which we express most generally in the expansion

$$C \rightarrow C_0 \left(1 + \alpha_0 \delta + \alpha_1 \delta^2 + \alpha_2 \delta^3 + \ldots\right)$$

- We define the momentum compaction factor as

$$\alpha_c = \frac{1}{C_0} \frac{dC}{d\delta} = \left(\alpha_0 + 2\alpha_1 \delta + 3\alpha_2 \delta^2 + \ldots\right) \equiv \frac{1}{\gamma_{tr}^2}$$

Longitudinal dynamics - drift

The difference in revolution frequency can be expressed as

$$\frac{\Delta\omega}{\omega_0} = \frac{\beta}{\beta_0}\frac{C_0}{C} - 1$$

Expanding the two factors as

$$\frac{\beta}{\beta_0} = f(\delta)\,, \quad \text{and} \quad \frac{C}{C_0} = g(\delta)$$

we obtain the phase slippage factor as

$$\frac{\Delta\omega}{\omega_0} = \frac{f(\delta)}{g(\delta)} - 1 = -\eta(\delta)\delta$$

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Convince yourself that:

$$f(\delta) = \frac{\delta+1}{\sqrt{\beta_0^2\delta^2+\beta_0^2\delta+1}}$$

We already know the expansion for this term:

$$(1 + \alpha_0\delta + \alpha_1\delta^2 + \ldots)$$

Longitudinal dynamics - drift

- The slippage factor is really a function of the transverse optics, the reference momentum and the deviation from the latter (γ_{tr}, p_0, δ). It can be expanded as

$$\frac{\Delta\omega}{\omega_0} = -\eta(\delta)\delta = -(\eta_0 + \eta_1\delta + \eta_2\delta^2 + \ldots)\delta$$

and expressed to lowest order in δ as

$$\eta \approx \eta_0 = \alpha_0 - \frac{1}{\gamma_0^2}$$

- The lowest order momentum compaction factor can be derived to

$$\alpha_0 = \frac{1}{C_0}\oint \frac{D(s)}{\rho}\,ds$$

See Hannes' lectures

Longitudinal dynamics - drift

With the understanding of phase slippage we can now write down the first part (drift) of the equations of motion in the longitudinal phase space (z, δ)

$$\dot{z} = -\eta\beta c\delta$$

This synchrotron drift would be generated by a Hamiltonian

$$H = -\frac{1}{2}\eta\beta c\,\delta^2$$

A consequency of this is the occurrence of transition crossing when the slippage factor switches sign.

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velocity can be positive or negative depending on whether particles are below or above transition

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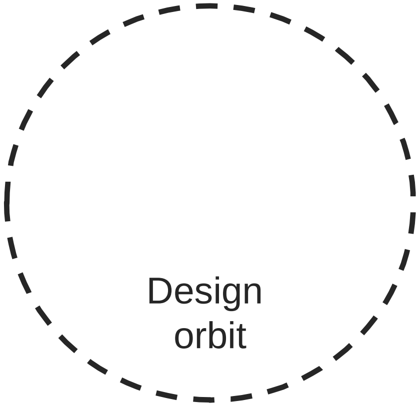
Longitudinal dynamics – transition

- The phase slippage is a result of the competing of the two effects:



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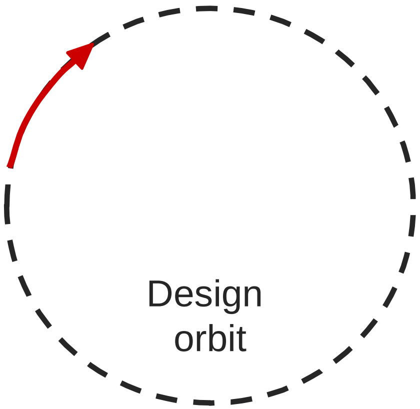

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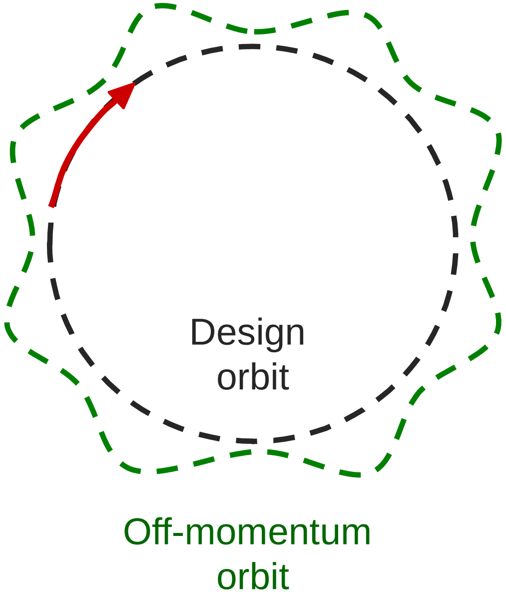
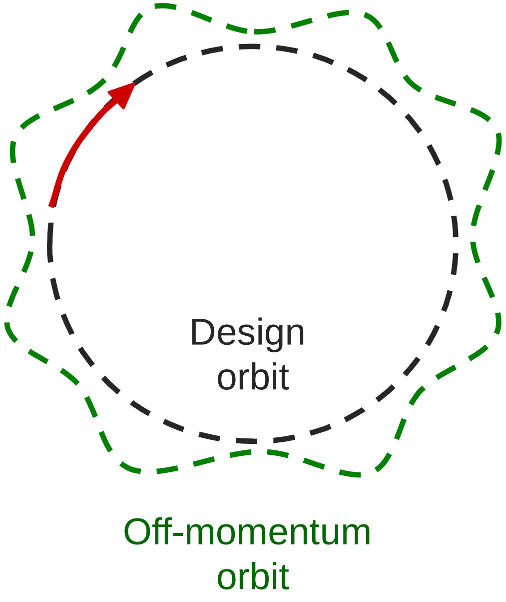
  - Gain in path length: i.e. high momentum particles travel on a longer path



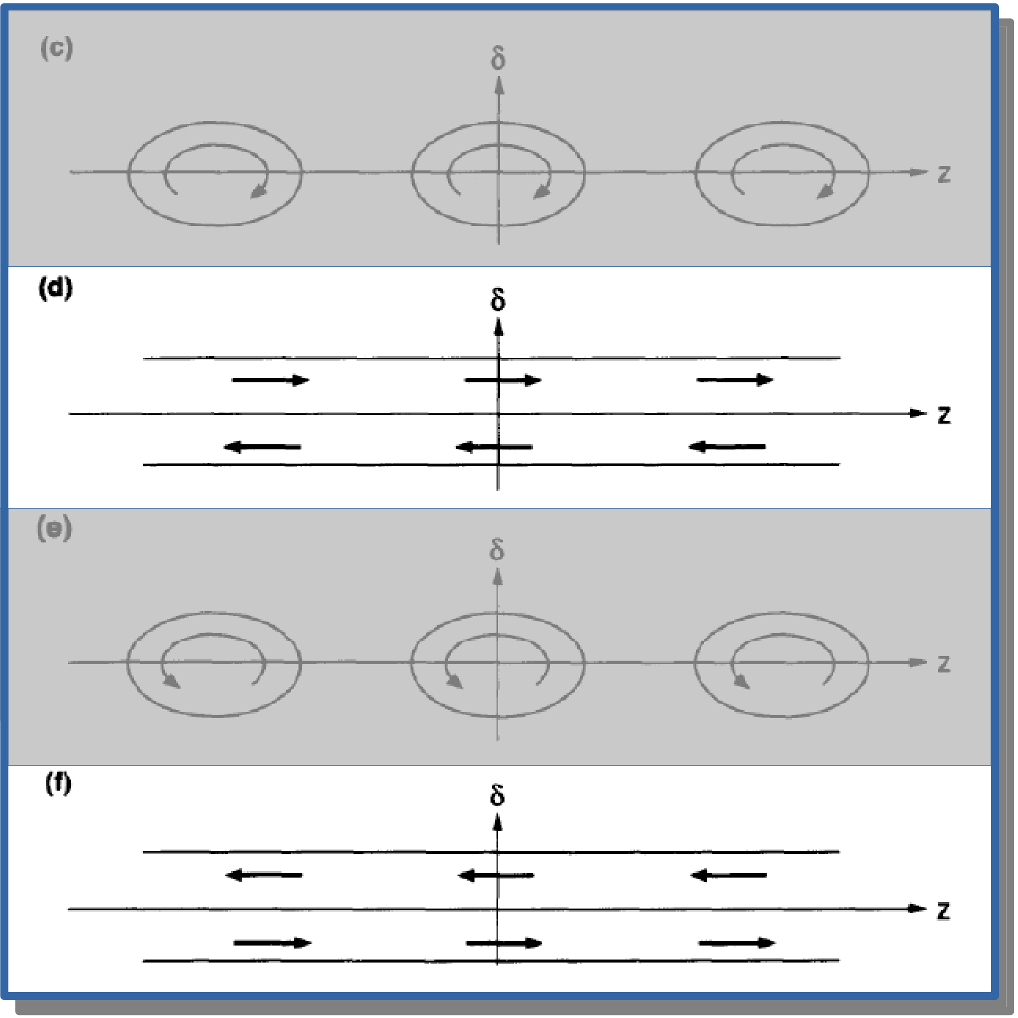
Longitudinal dynamics – transition

- The phase slippage is a result of the competing of the two effects:
  - Gain in velocity: i.e. high momentum particles move faster
  - Gain in path length: i.e. high momentum particles travel on a longer path
  - Below transition, the first effect wins: high momentum particles will advance compared to low momentum particles
  - Above transition, the second effect dominates: high momentum particles will fall back compared to low momentum particles



Longitudinal dynamics – transition

- The synchronous particle has zero momentum offset and always takes T0 to go around, i.e. is always observed with z=0.
- Particles with positive z arrive earlier at the observer (negative delay t), those with negative z arrive later (positive delay t)
- Bunched beams: below transition particles are focused back by deceleration. Opposite above transition.
- Coasting beams: below transition particles with positive momentum offset shear toward positive z (absence of focusing). Opposite above transition



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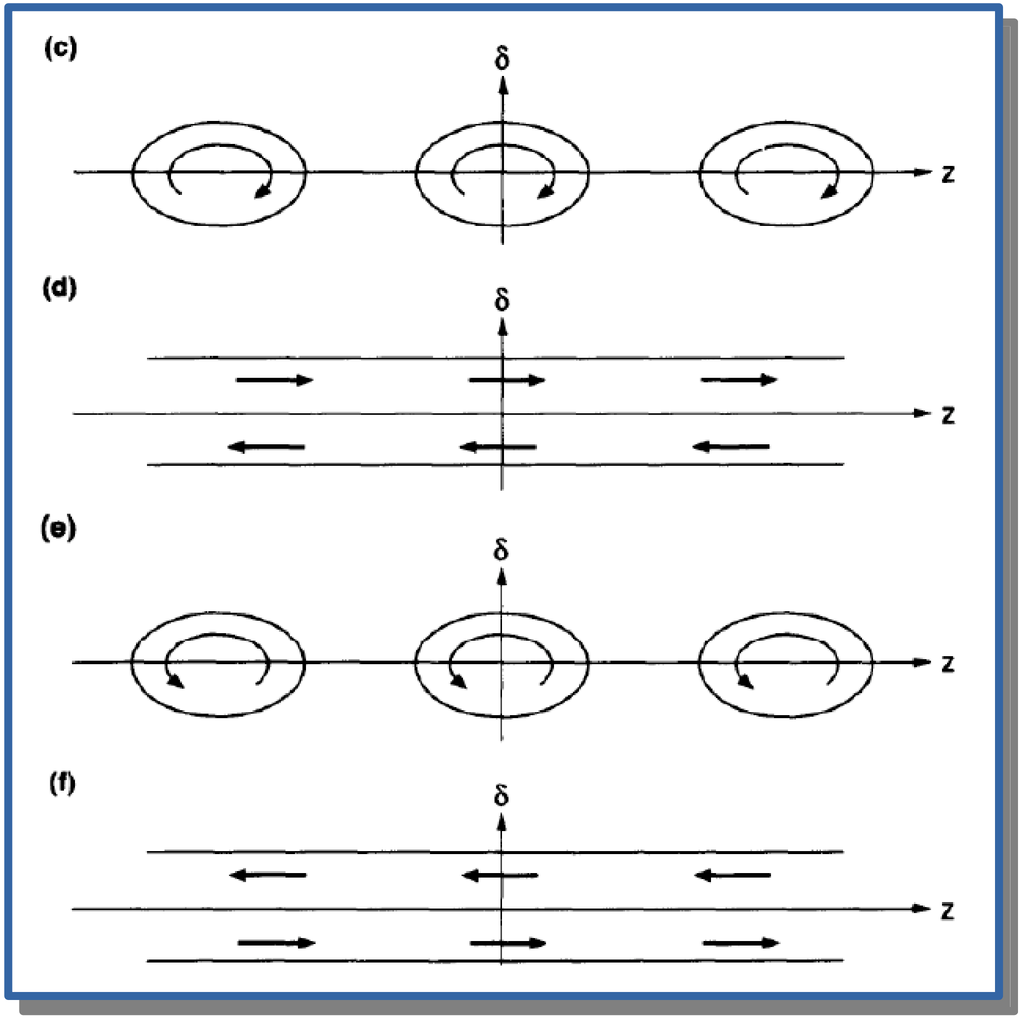

The role of the RF - kick

The role of the RF can be decomposed in two main functions:

Acceleration: to ensure that the beam remains on the design orbit. For this, it needs to compensate a change in the bending fields by accelerating the beam adequately to keep the orbit fixed

$$\rho = \frac{\Delta p_0}{e\Delta B}$$

Focussing: to ensure that the beam is longitudinally focused in bunches. For this, particles lagging behind should experience an additional acceleration whereas particles ahead should experience a reduced acceleration (and inverse when above transition).

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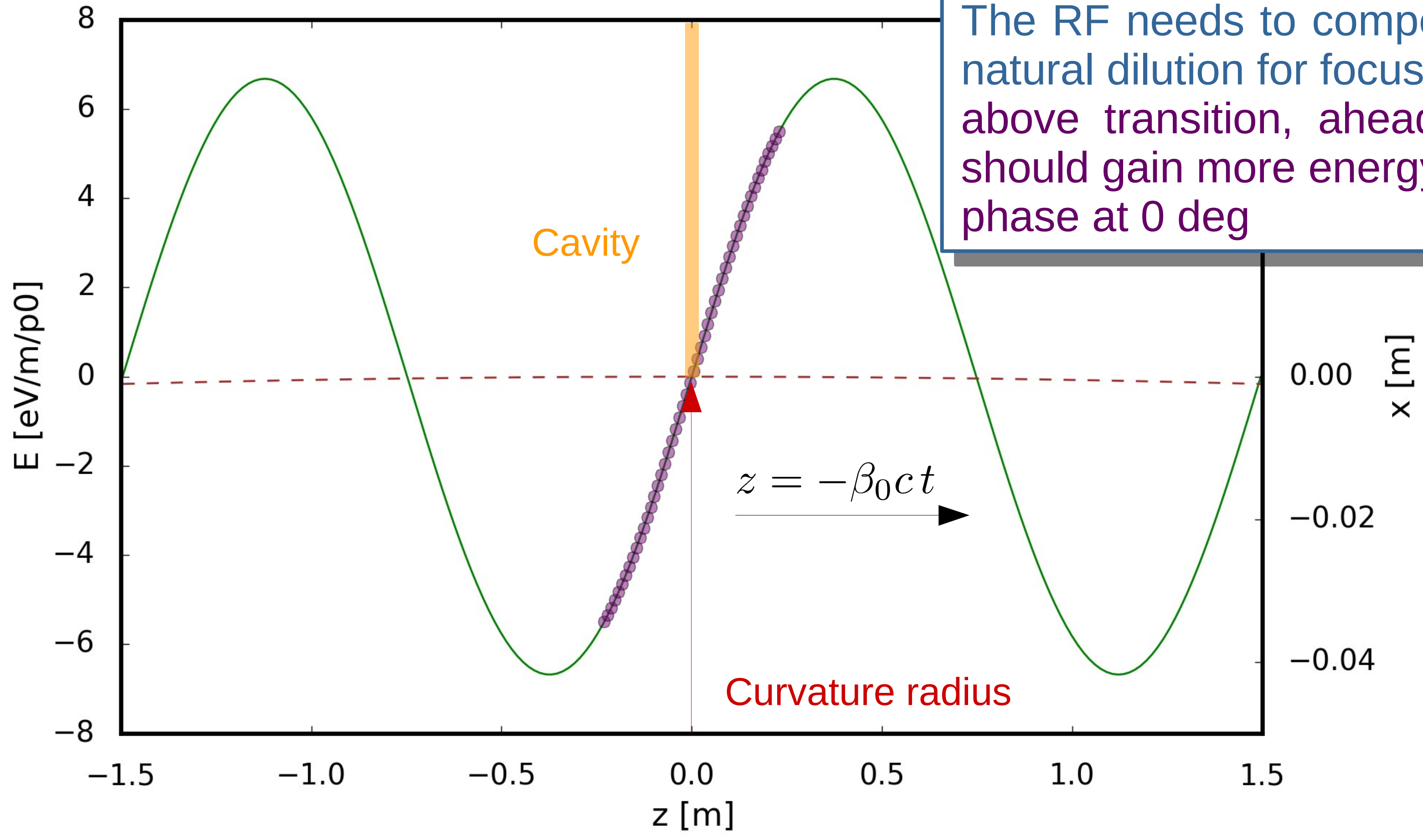

The RF needs to compensate the natural dilution for focusing:
above transition, ahead particles should gain more energy → stable phase at 0 deg

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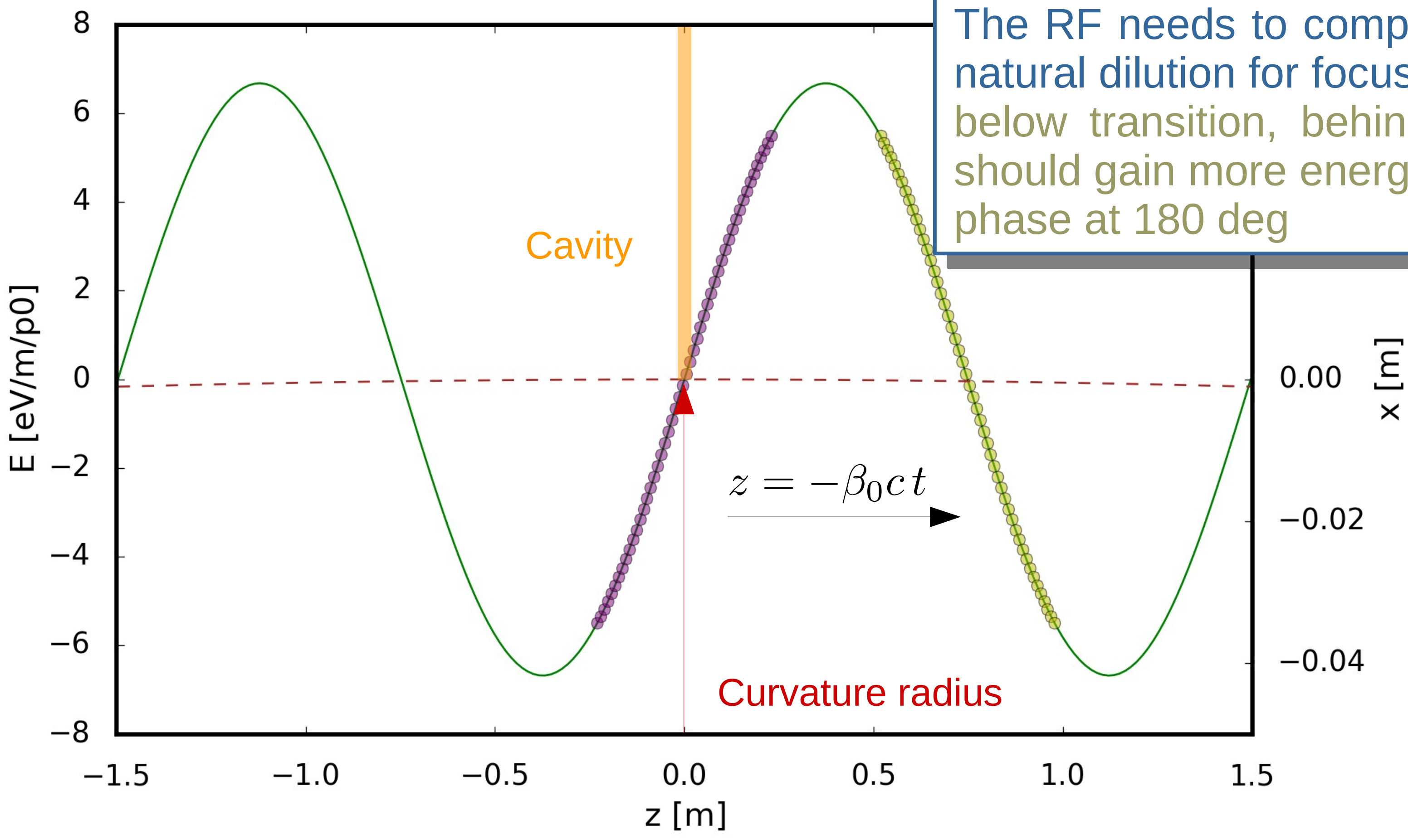

The RF needs to compensate the natural dilution for focusing:
below transition, behind particles should gain more energy → stable phase at 180 deg

The role of the RF - kick

Acceleration and focusing can be formalised. The average energy gain per unit length from the RF field can be written as

$$E = \frac{eV}{C} \sin\left(\frac{hz}{R} + \varphi\right)$$

Acceleration requires that part of the RF field is used to provide an energy gain such that per revolution

$$p_0 \to p_1 = p_0 + \frac{\Delta E}{\beta c}$$

The remaining field is used for focusing and give rise to the synchroton motion

$$\delta_0 \to \delta_1 = \delta_0 + \frac{eVT_0}{p_0 C} \sin\left(\frac{hz}{R} + \varphi\right) - \frac{\Delta E}{p_0 \beta c}$$

The role of the RF - kick

This synchrotron kick would be generated by a Hamiltonian

$$H = \frac{eV}{2\pi p_0 h} \cos\left(\frac{hz}{R} + \varphi\right) + \frac{\Delta E}{p_0 C} z$$

The synchrotron Hamiltonian

With the considerations made in the previous slides we now postulate the synchrotron Hamilonian as

$$H = -\frac{1}{2}\eta\beta c\,\delta^2 + \frac{e}{p_0 C}\,V(z)$$

with the most general voltage provided for synchrotron motion by the RF

$$V(z) = \sum_i \frac{V_i R}{h_i}\cos\left(\frac{h_i z}{R} + \varphi_i\right) + \frac{\Delta E}{e}\,z$$

The synchrotron Hamiltonian

Often it is more convenient to set the zero level of the Hamiltonian to a more meaningful point z_c, i.e.

$$H(z,\delta) = -\frac{1}{2}\eta\beta c\,\delta^2 + \frac{eV}{2\pi p_0 h}\left(\cos\left(\frac{hz}{R}\right) - \cos\left(\frac{hz_c}{R}\right) + \left(\frac{hz}{R} - \frac{hz_c}{R}\right)\frac{\Delta E}{eV}\right)$$

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The equations of motion remain invariant as

$$\begin{pmatrix} \dot{z} \\ \dot{\delta} \end{pmatrix} = J\vec{\nabla}H = \begin{pmatrix} \frac{\partial H}{\partial \delta} \\ -\frac{\partial H}{\partial z} \end{pmatrix} = \begin{pmatrix} -\eta\beta c\,\delta \\ \frac{eV}{p_0 C}\left(\sin\left(\frac{hz}{R}\right) - \frac{\Delta E}{eV}\right) \end{pmatrix},$$

J is the symplectic structure matrix

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Acceleration by ΔE with a single RF gives rise to the synchroneous position (synchroneous phase) with respect to the reference position (zero crossing of the fundamental RF), for which there is no synchrotron motion

$$\frac{hz_s}{R} = \phi_s = \arcsin\left(\frac{\Delta E}{eV}\right)$$

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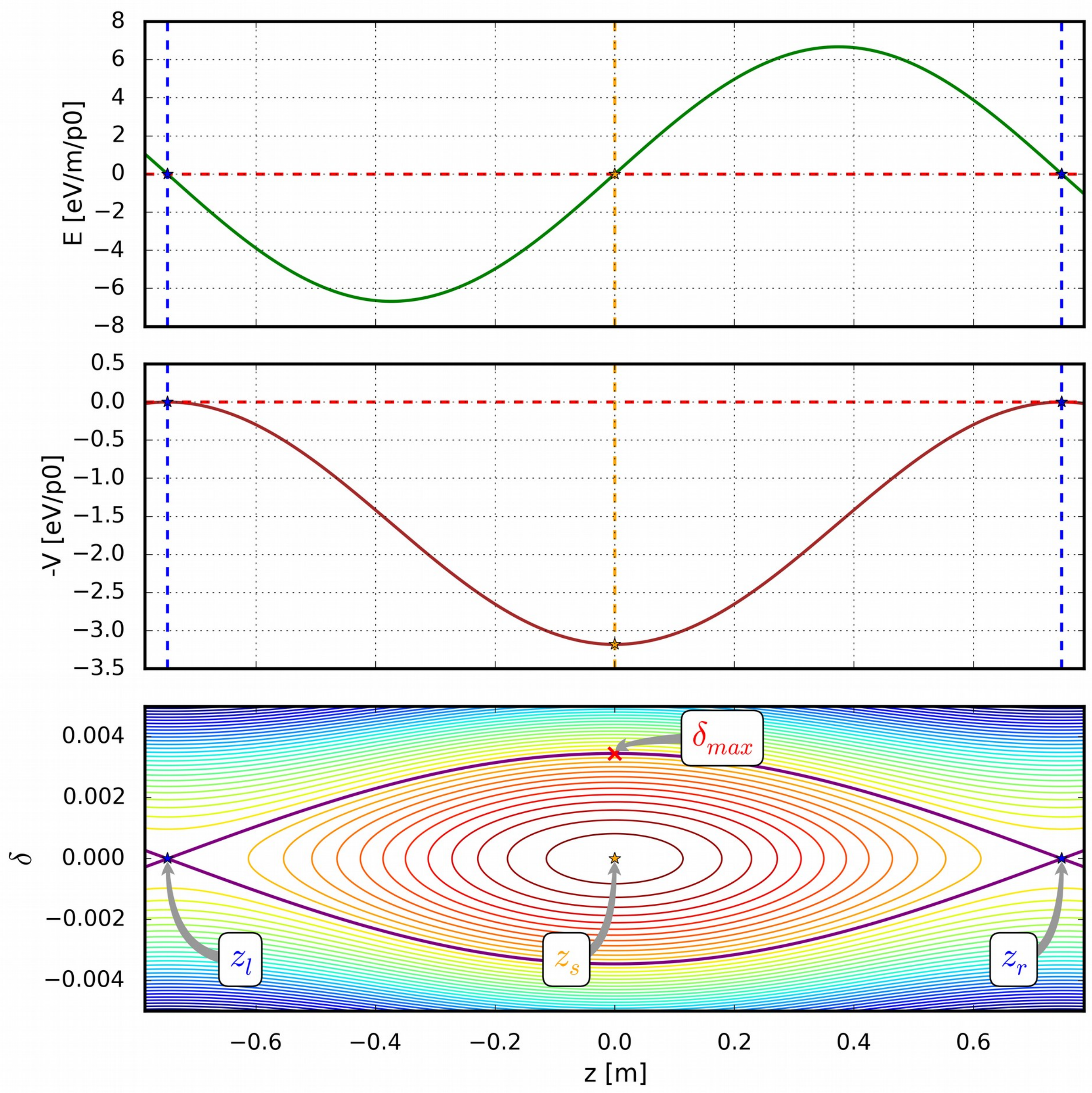
Let's now take a look at some examples of different RF buckets.

Single stationary bucket

$$z_s = 0$$

$$z_l = -\frac{\pi R}{h}$$

$$z_r = \frac{\pi R}{h} - z_s = \frac{\pi R}{h}$$



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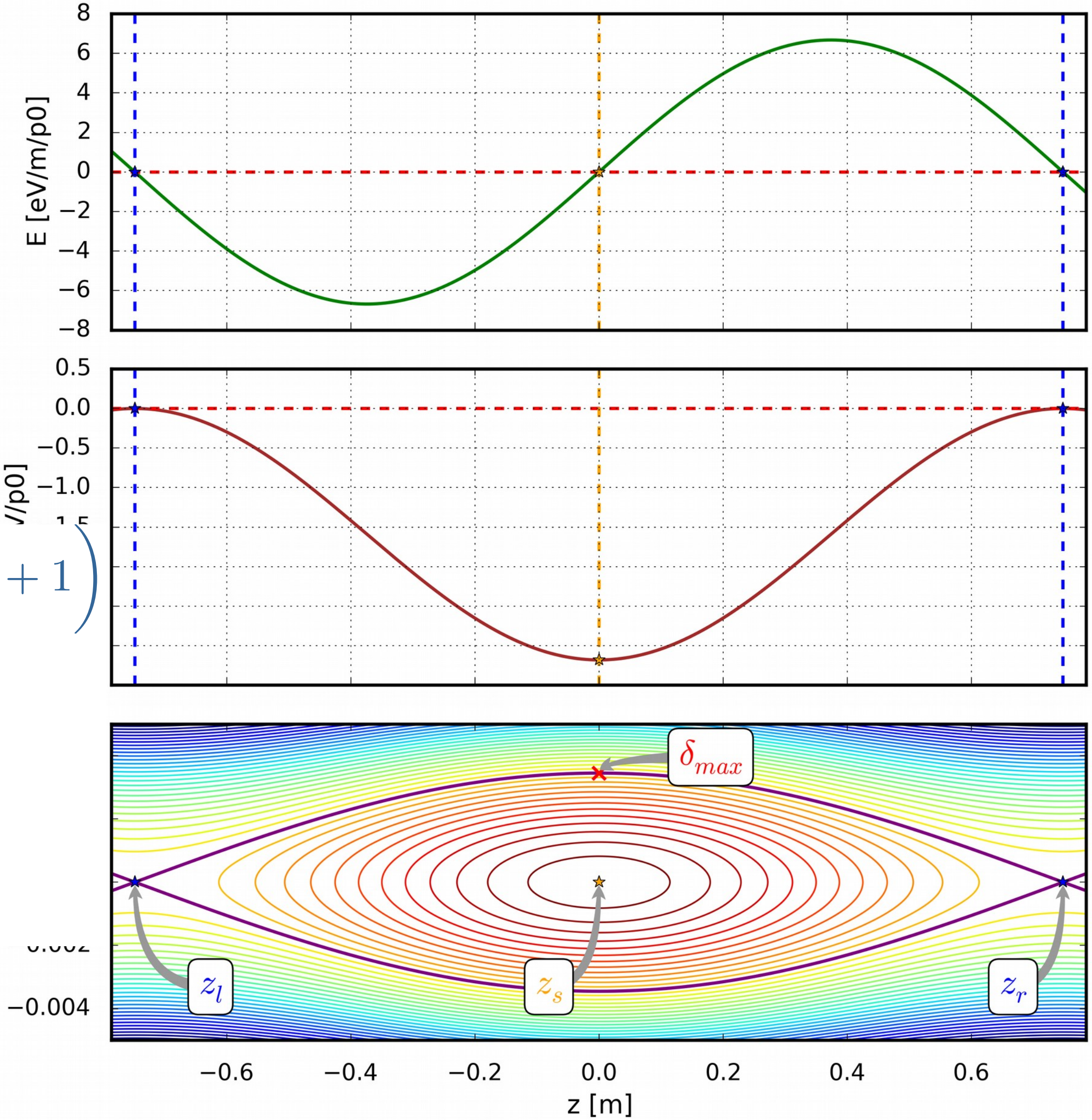

$$H(z,\delta) = -\frac{1}{2}\eta\beta c\,\delta^2 + \frac{eV}{2\pi p_0 h}\left(\cos\left(\frac{hz}{R}\right) + 1\right)$$

$$\delta_{\max} = \sqrt{\frac{2eV}{\pi\beta_0^2 E_0 |\eta| h}}$$

$$\mathcal{A} = \frac{16}{h}\sqrt{\frac{eV\beta_0^2 E_0}{2\pi\omega_0^2 |\eta| h}}$$

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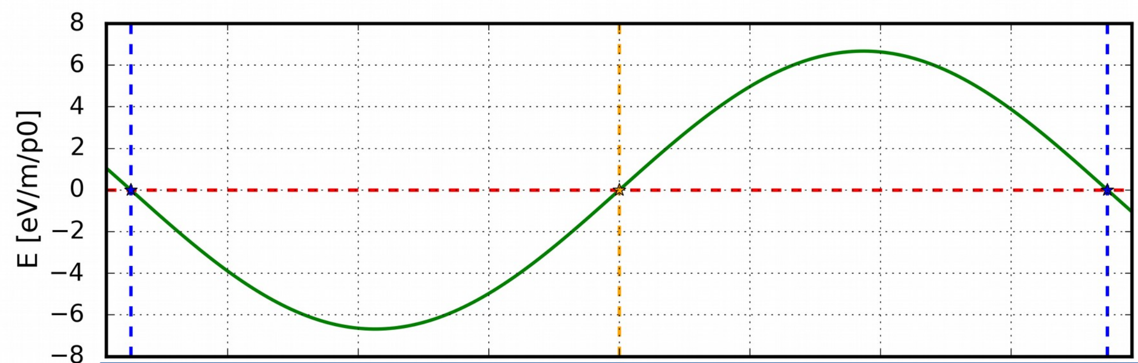

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For more general cases, we need to solve

- $E(z) = 0 \Rightarrow z_s, z_{l/r}$
- $V(z) = 0 \Rightarrow z_{r/l}$
- $\delta_{\text{separatrix}}(z) : H(z,\delta) = 0$
- $\delta_{\max} = \delta_{\text{separatrix}}(z_s)$
- $\mathcal{A} = 2p_0 \int_{z_l}^{z_r} \delta_{\text{separatrix}}(z)\, dz$

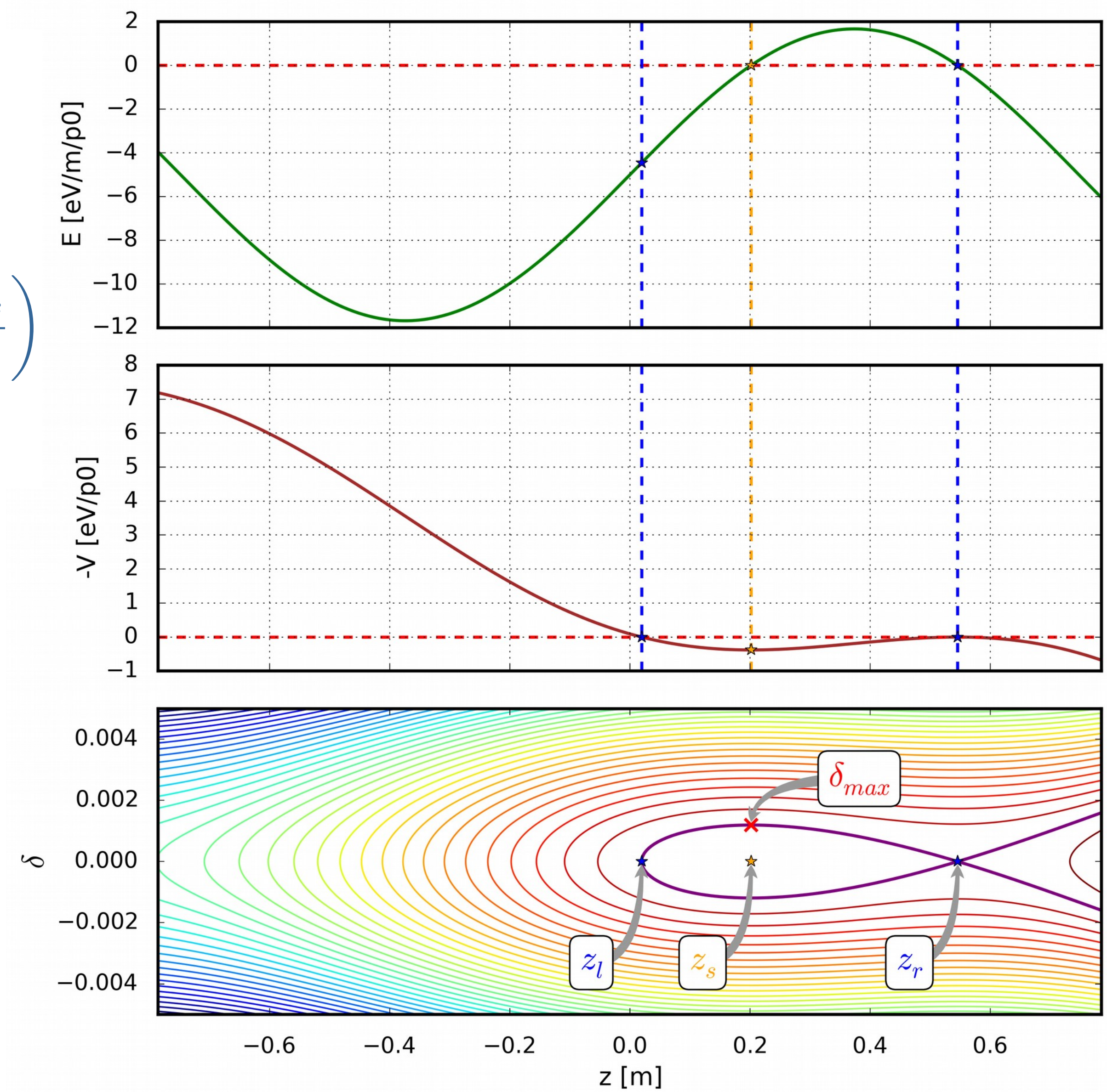
Single accelerating bucket

$$z_s = \frac{R}{h}\arcsin\left(\frac{\Delta E}{eV}\right)$$

$$z_l : \cos\left(\frac{hz_l}{R}\right) - \frac{hz_l}{R}\sin\left(\frac{hz_s}{R}\right)$$

$$= \left(\frac{hz_s}{R} - \pi\right)\sin\left(\frac{hz_s}{R}\right) - \cos\left(\frac{hz_s}{R}\right)$$

$$z_r = \frac{\pi R}{h} - z_s$$



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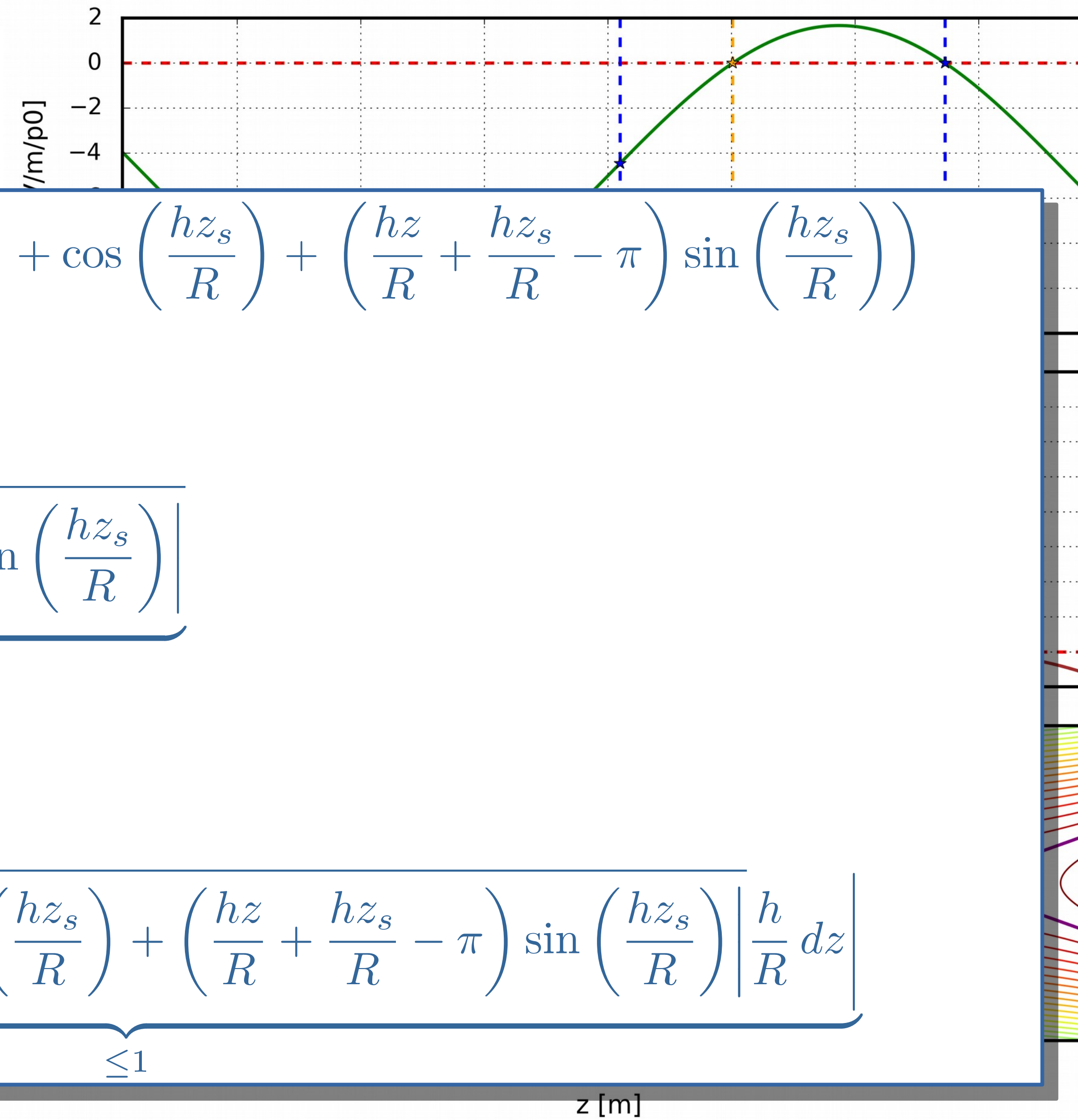


$$H(z,\delta) = -\frac{1}{2}\eta\beta c\,\delta^2 + \frac{eV}{2\pi p_0 h}\left(\cos\left(\frac{hz}{R}\right) + \cos\left(\frac{hz_s}{R}\right) + \left(\frac{hz}{R} + \frac{hz_s}{R} - \pi\right)\sin\left(\frac{hz_s}{R}\right)\right)$$

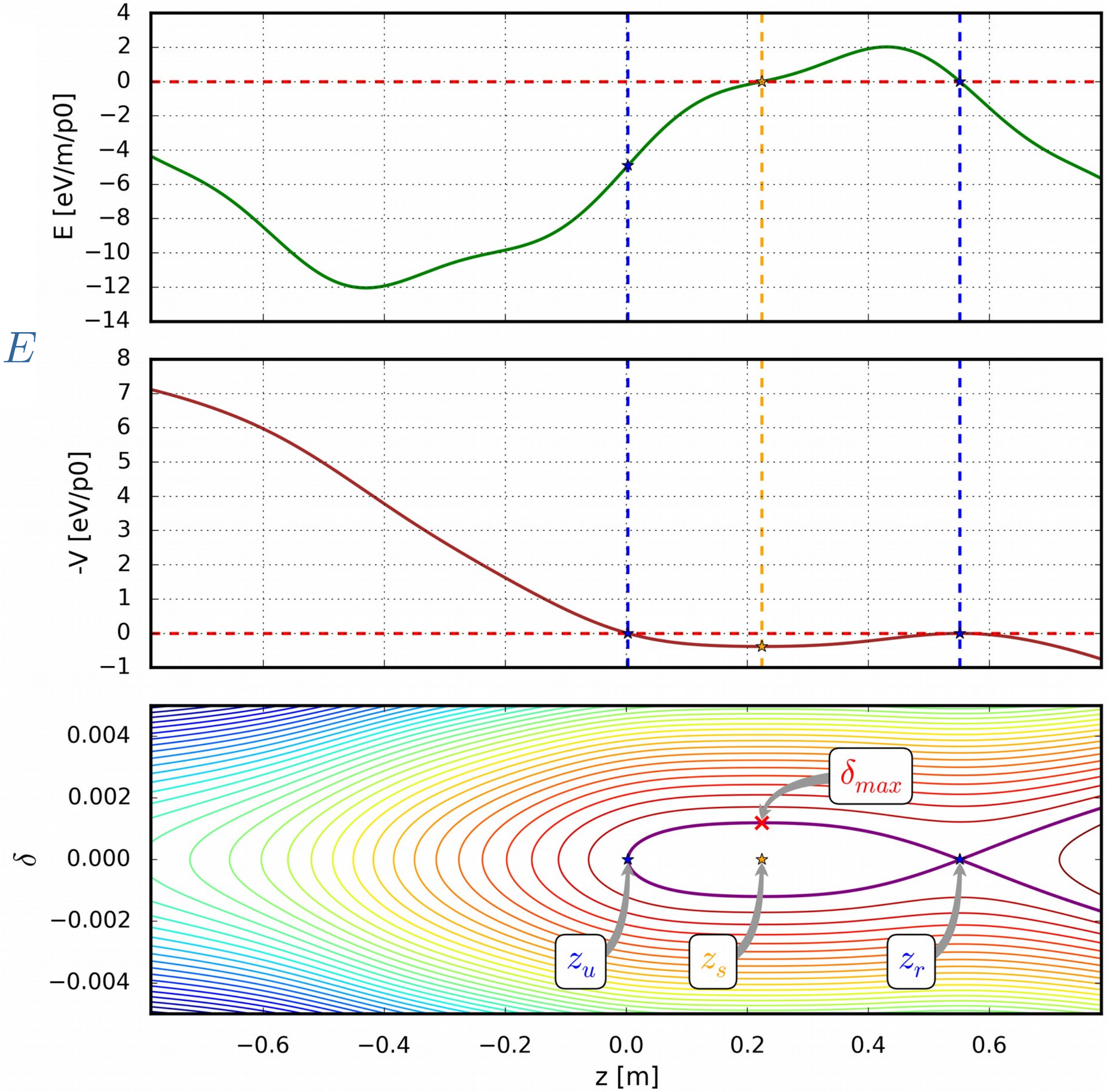
$$\delta_{\max} = \sqrt{\frac{2eV}{\pi\beta_0^2 E_0|\eta|h}} \times \underbrace{\sqrt{\left|\cos\left(\frac{hz_s}{R}\right) + \left(\frac{hz_s}{R} - \frac{\pi}{2}\right)\sin\left(\frac{hz_s}{R}\right)\right|}}_{\leq 1}$$

$$\mathcal{A} = \frac{16}{h}\sqrt{\frac{eV\beta_0^2 E_0}{2\pi\omega_0^2|\eta|h}} \times \underbrace{\left|\frac{1}{4\sqrt{2}}\int_{z_l}^{z_r}\sqrt{\left|\cos\left(\frac{hz}{R}\right) + \cos\left(\frac{hz_s}{R}\right) + \left(\frac{hz}{R} + \frac{hz_s}{R} - \pi\right)\sin\left(\frac{hz_s}{R}\right)\right|}\,\frac{h}{R}\,dz\right|}_{\leq 1}$$

Multiple accelerating bucket

$$z_s, z_r : \sum_i eV_i \sin\left(\frac{h_i z}{R} + \varphi_i\right) = \Delta E$$

$$z_l : \sum_i \frac{eV_i R}{h_i}\left(\cos\left(\frac{h_i z}{R} + \varphi_i\right) - \cos\left(\frac{h_i z_r}{R} + \varphi_i\right)\right) = (z_r - z)\Delta E$$



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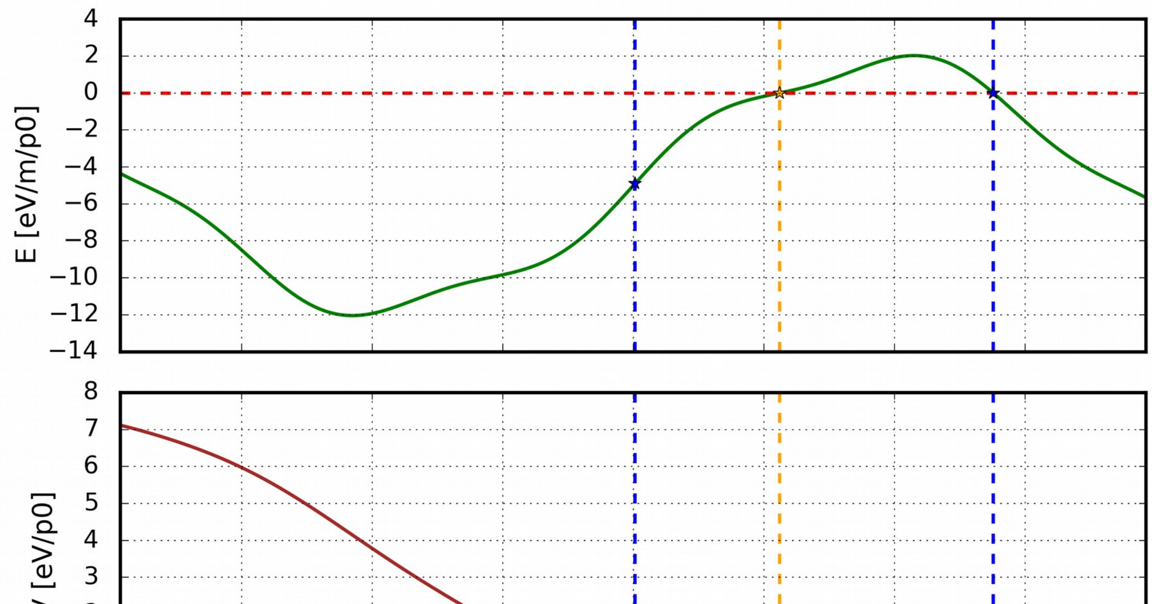

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$$\delta_{\mathrm{max}} = \sqrt{\frac{e}{\pi\beta_0^2 E_0|\eta|R}\left(\sum_i \frac{V_i R}{h_i}\left(\cos\left(\frac{h_i z_s}{R} + \varphi_i\right) - \cos\left(\frac{h_i z_r}{R} + \varphi_i\right)\right) + (z_s - z_r)\frac{\Delta E}{e}\right)}$$

$$\mathcal{A} = 2p_0 \int_{z_l}^{z_r} \delta_{\mathrm{separatrix}}\, dz$$

Small amplitude approximation

For particles close to the synchroneous phase we may linearize the potential:

$$
\begin{aligned}
H(z,\delta) &= -\frac{1}{2}\eta\beta c\,\delta^2 \\
&\quad + \frac{eV}{2\pi p_0 h}\left(\cos\left(\frac{hz}{R}\right) + \cos\left(\frac{hz_s}{R}\right) + \left(\frac{hz}{R} + \frac{hz_s}{R} - \pi\right)\sin\left(\frac{hz_s}{R}\right)\right) \\
&\approx -\frac{1}{2}\eta\beta c\,\delta^2 - \frac{eVh}{4\pi p_0 R^2}\cos(\phi_s)\,z^2
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This allows the definition of the (small amplitude) synchrotron tune

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z'' + \left(\frac{\omega_s}{\beta c}\right)^2 z = 0\,, \quad \omega_s^2 = \frac{eV\eta h\beta c}{2\pi p_0 R^2}\cos(\phi_s)
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$$

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Qs = \frac{\omega_s}{\omega_0} = \sqrt{\frac{eV\eta h}{2\pi E_0 \beta^2}\cos(\phi_s)}
$$

Rediscover the stability criterion
$\phi = \pi$ below transition ($\eta < 0$)
$\phi = 0$ above transition ($\eta > 0$)

Matching

- For an ensemble of particles it is important to be matched to prevent filamentation, i.e. emittance increase.

- We will learn that, given a single particle probability density function ψ to describe the particle ensemble, the ensemble is matched if $\psi = \psi(H)$.

- Given the single particle Hamiltonian

$$H(z,\delta) = -\frac{1}{2}\eta\beta c\,\delta^2 - \frac{eVh}{4\pi p_0 R^2}\cos(\phi_s)\,z^2\,,$$

let's assume a Gaussian PDF as

$$\psi(H) \propto \exp\left(\frac{H}{H_0}\right) = \exp\left(-\frac{\eta\beta c\,\delta^2}{2H_0}\right)\exp\left(-\frac{eVh\cos(\phi_s)\,z^2}{4\pi p_0 R^2 H_0}\right)\,.$$

Matching

It follows that

- $\sigma_\delta^2 = \dfrac{H_0}{\eta\beta c}$
- $\sigma_z^2 = \dfrac{2\pi p_0 R^2 H_0}{eVh\cos(\phi_s)}$

and, hence, with $Q_s^2 = \frac{eV\eta h}{2\pi\beta^2 E}\cos(\phi_s)$ the matching condition:

$$\sigma_z = \frac{\eta R}{Q_s}\sigma_\delta \equiv \beta_z\sigma_\delta\,.$$

Just as in the transvere plane, a Gaussian beam with the correct ratio σ_z/σ_δ compared to the external focusing given by $1/\beta_z$ will not perform any quadrupolar oscillations or suffer emittance increase due to filamentation.

Matching

The (normalized) longitudinal emittance is typically defined as the 2σ-area of the rms emittance

$$\varepsilon_z^{(n)} = 4\pi\sqrt{\langle z^2\rangle\langle dp^2\rangle - \langle z\,dp\rangle^2} \quad [\mathrm{eV\,s}]\,.$$

For a bi-Gaussian distribution, this simplifies to

$$\varepsilon_z^{(n)} = 4\pi\sigma_z\sigma_{dp}$$

and corresponds to the 2σ-area occupied by the beam in phase space. Hence, for a matched bi-Gaussian distribution, we can finally write

$$\begin{aligned}\varepsilon_z^{(n)} &= 4\pi\frac{Q_s}{\eta R}p_0\sigma_z^2\\ &= 4\pi\frac{\eta R}{Q_s p_0}\sigma_{dp}^2\,.\end{aligned}$$

Matching

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For a bi-Gauss

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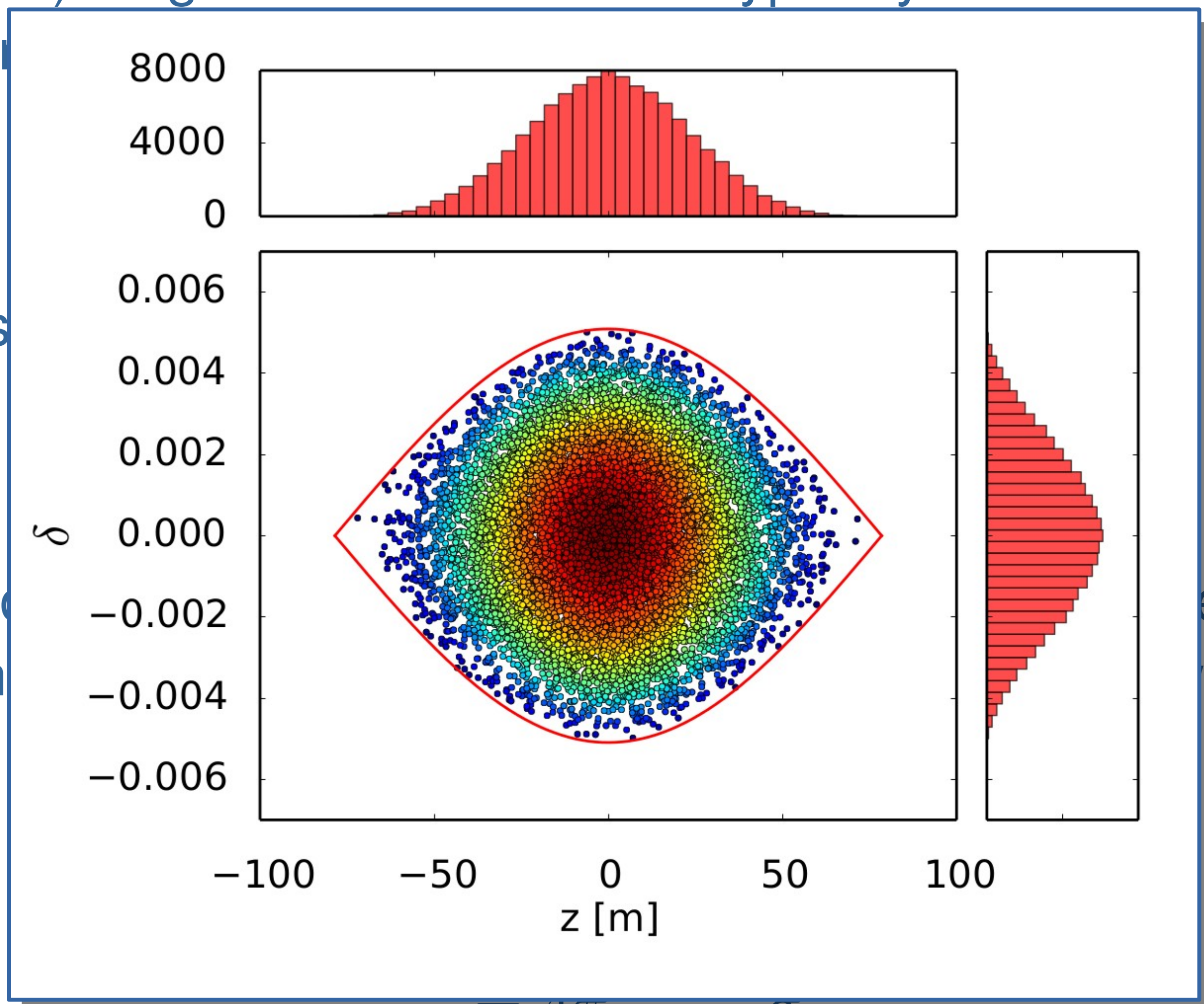

$$= 4\pi \frac{\quad}{Q_s} \sigma_{dp} .$$

Coupling to the transverse plane

Quantities such as the chromaticity couple the longitudinal and the transverse plane. Computing macroscopic quantities out of multiparticle distributions, thus, becomes challenging, even without introducing collective effects. For example, the mean horizontal centroid position can be expressed as

$$
\begin{aligned}
\langle u \rangle (t) &= \int u \, \rho(u, u', z, \delta, t) \, du \, du' \, dz \, d\delta \\
&= \int u(\hat{u}, \hat{u}', \hat{z}, \hat{\delta}, t) \, \rho(\hat{u}, \hat{u}', \hat{z}, \hat{\delta}) \, d\hat{u} \, d\hat{u}' \, d\hat{z} \, d\hat{\delta} \, ,
\end{aligned}
$$

where, now, the single particle time evolution is given as

$$
u(\hat{u}, \hat{u}', t) = \hat{u} \cos\Big(\omega_u + \Delta\omega_u(J_u, \delta(t))\Big) + \beta_u \hat{u}' \sin\Big(\omega_u + \Delta\omega_u(J_u, \delta(t))\Big) \, .
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Modulation of the transverse frequency by the synchrotron frequency

Summary

- Reference coordinates in the longitudinal plane
- The drift equation of motion
- Transition
- The kick equation of motion
- The synchrotron Hamiltonian
- Small amplitude approximation
- Multi-particle peculiarities

THE END