CERN

U.S. Particle Accelerator School
Education in Beam Physics and Accelerator Technology

Collective effects in Beam Dynamics

Giovanni Rumolo, Hannes Bartosik and Kevin Li

USPAS, one-week course, 19-24 January, 2015
http://uspas.fnal.gov/index.shtml

Outline

1. Introduction
    - Collective effects
    - Transverse single particle dynamics including systems of many non-interacting particles
    - Longitudinal single particle dynamics including systems of many non-interacting particles

2. Space charge
    - Direct space charge (transverse)
    - Indirect space charge (transverse)
    - Longitudinal space charge

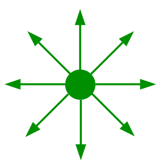
Outline

1. Introduction
   - Collective effects
   - Transverse single particle dynamics including systems of many non-interacting particles
   - Longitudinal single particle dynamics including systems of many non-interacting particles

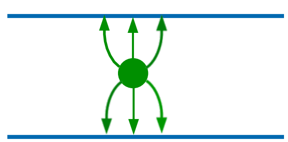
2. Space charge
   - Direct space charge (transverse)
   - Indirect space charge (transverse)
   - Longitudinal space charge

Space Charge

- A beam of charged particles induces electromagnetic fields when circulating inside the vacuum chamber.
- These self induced fields depend on
  - Beam current (intensity) and particle distribution
  - Geometry and material of the surrounding vacuum chamber and machine elements
- When we talk about space charge we think about



direct space charge:
interaction of charged particles in free space



indirect space charge:
interaction with image charges and currents induced in perfect conducting walls and ferromagnetic materials close to the beam pipe

- Space charge effects
  - cause tune shifts (transverse and longitudinal)
  - can result in longitudinal instability (negative mass instability)

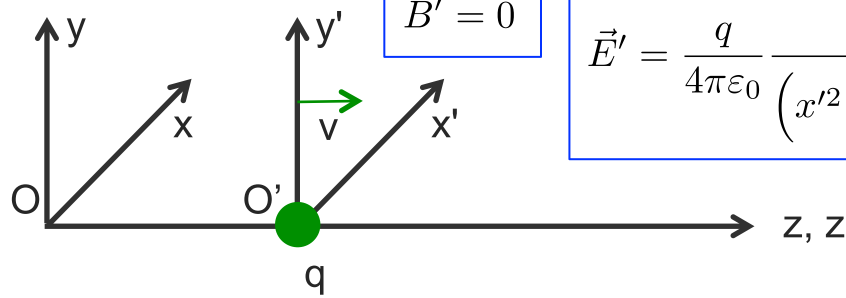
Point charge with constant velocity

- Consider point charge q with velocity *v* in z direction
  - Rest frame coordinate system with origin O
  - Moving coordinate system with origin O' where charge is at rest

  } they coincide at t=0

- Relativistic transforms of the coordinates and fields

$$\begin{cases} x' = x \\ y' = y \\ z' = \gamma(z - vt) \end{cases}$$



$$\vec{B}' = \vec{0}$$

$$\vec{E}' = \frac{q}{4\pi\varepsilon_0} \frac{1}{\left(x'^2 + y'^2 + z'^2\right)^{3/2}} \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}$$

Relativistic transforms of fields

$$\begin{cases} E_x = \gamma\left(E'_x + vB'_y\right) \\ E_y = \gamma\left(E'_y - vB'_x\right) \\ E_z = E'_z \end{cases} \quad \begin{cases} B_x = \gamma\left(B'_x - vE'_y/c^2\right) \\ B_y = \gamma\left(B'_y + vE'_x/c^2\right) \\ B_z = B'_z \end{cases}$$

Fields in the rest frame at t=0

$$\vec{E} = \frac{q}{4\pi\varepsilon_0} \frac{\gamma}{\left(x^2 + y^2 + \gamma^2 z^2\right)^{3/2}} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

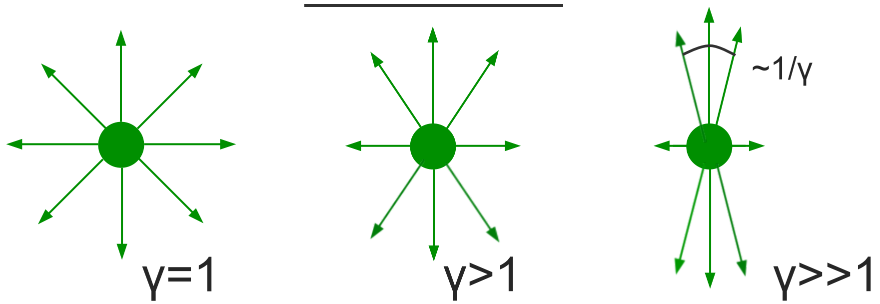
$$B_z = 0$$
$$B_x = -vE_y/c^2$$
$$B_y = +vE_x/c^2$$

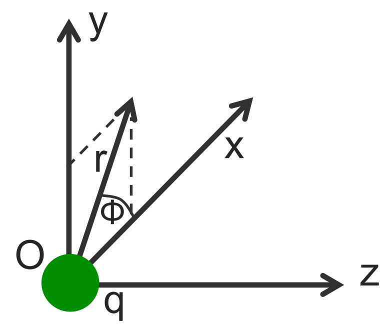
… electric field has no spherical symmetry in rest frame

Point charge with constant velocity

- In the rest frame O
  - Electric field lost spherical symmetry but still “symmetry with respect the z-axis”
  - Magnetic field is transverse to the particle motion

Electric field





$$E_x(z=t=0) = \frac{q}{4\pi\varepsilon_0}\frac{\gamma x}{\left(x^2+y^2\right)^{3/2}}$$

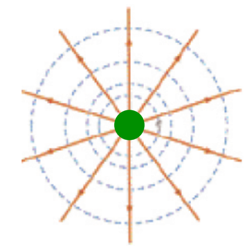
$$E_y(z=t=0) = \frac{q}{4\pi\varepsilon_0}\frac{\gamma y}{\left(x^2+y^2\right)^{3/2}}$$

$$E_z(x=y=t=0) = \frac{q}{4\pi\varepsilon_0}\frac{1}{\gamma^2 z^2}$$

for *z=vt*

$$E_r = \frac{q}{4\pi\varepsilon_0}\frac{\gamma}{r^2}$$

Magnetic field



$$B_z = 0$$

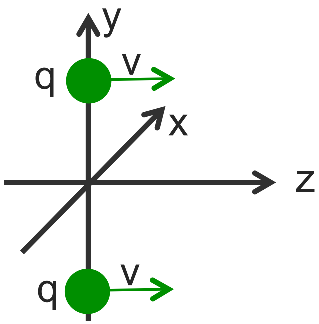
$$B_x = -vE_y/c^2$$

$$B_y = +vE_x/c^2$$

$$B_\phi = \frac{\beta E_r}{c}$$

Two point charges with same velocity

- Consider two point charges with the same charge q and with same velocity $v_1 = v_2 = v$ on parallel trajectories
  - In the rest frame, we know already the electric and magnetic fields generated by a "source" particle
  - The force on the "test" particle is given by the Lorentz force
- The attractive magnetic force tends to compensate the repulsive electric force



Fields generated by a source particle with velocity v

$$E_r = \frac{q}{4\pi\varepsilon_0}\frac{\gamma}{r^2}$$

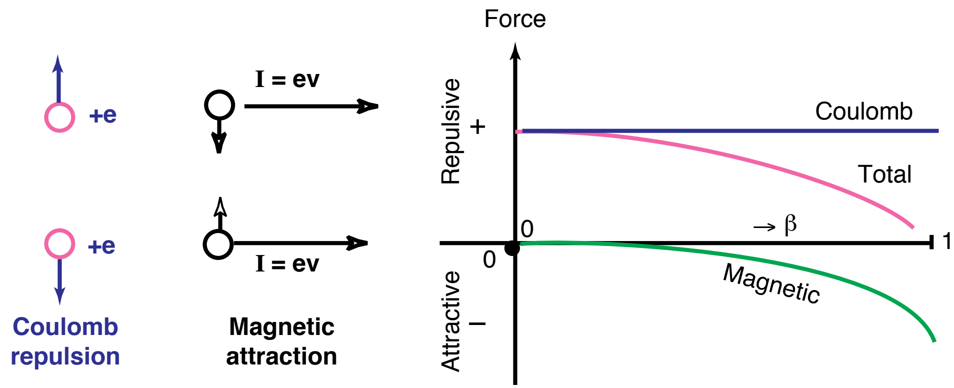
$$B_\phi = \frac{\beta E_r}{c}$$

Lorentz force acting on the test particle

$$F_r = q\left(E_r - vB_\phi\right) = q\left(E_r - \beta^2 E_r\right) = \frac{qE_r}{\gamma^2} = \frac{qq}{4\pi\varepsilon_0\gamma r^2}$$

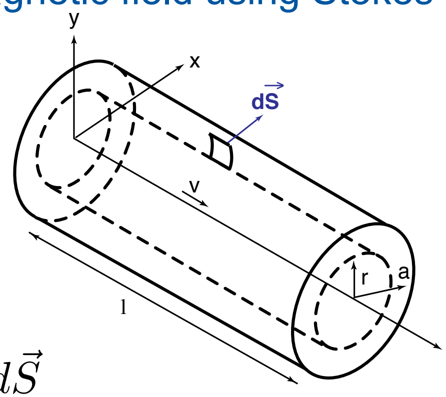
Two point charges with same velocity

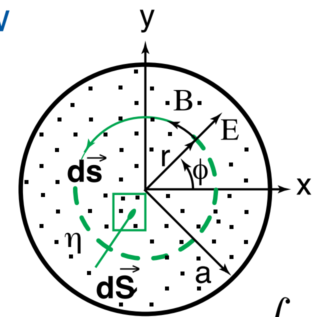
- Consider two point charges with the same charge q and with same velocity $v_1=v_2=v$ on parallel trajectories
  - In the rest frame, we know already the electric and magnetic fields generated by a "source" particle.
  - The force on the "test" particle is given by the Lorentz force
- The attractive magnetic force tends to compensate the repulsive electric force
  - At rest the two particles experience only the repulsive Coulomb force
  - When travelling with velocity v the particles represent two parallel currents which attract each other by the induced magnetic field
  - The forces become equal at the speed of light and thus cancel



Coasting beam with uniform charge density

- Assume a coasting beam of circular cross section with radius *a* and uniform charge density $\eta=\lambda/\pi a^2$ [Cb/m³] moving at constant velocity $v=\beta c$
  - Calculate the electric field using Gauss' law
  - Calculate the magnetic field using Stokes' law





Maxwell equation

$$\vec{\nabla}\cdot\vec{E}=\frac{\eta}{\varepsilon_0}$$

Gauss' law

$$\iiint \vec{\nabla}\cdot\vec{E}\,dV=\iint \vec{E}\,d\vec{S}$$

$$\Rightarrow \quad \pi l r^2\frac{\eta}{\varepsilon_0}=2\pi l r E_r$$

due to symmetry the electric field has only a radial component

with line density $\lambda=\pi a^2\eta$

$$E_r=\frac{\lambda}{2\pi\epsilon_0}\frac{r}{a^2}$$

(for *r* < *a*)

Maxwell equation

$$\vec{\nabla}\times\vec{B}=\mu_0\vec{J}$$

Stokes' law

$$\oint \vec{B}\,d\vec{s}=\iint \vec{\nabla}\times\vec{B}\,d\vec{S}$$

$$\Rightarrow \quad 2\pi r B_\phi=\mu_0\pi r^2 J$$

with current density $J=\beta c\eta=\beta c\lambda/\pi a^2$

and $\mu_0=1/\varepsilon_0 c^2$

$$B_\phi=\frac{\lambda\beta}{2\pi\varepsilon_0 c}\frac{r}{a^2}$$

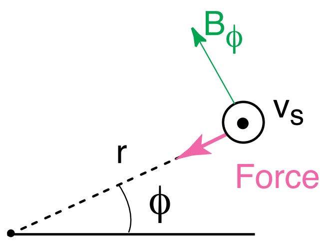
(for *r* < *a*)

Coasting beam with uniform charge density

- Assume a coasting beam of circular cross section with radius *a* and uniform charge density $\eta=\lambda/\pi a^2$ [Cb/m^3] moving at constant velocity $v=\beta c$
- Calculate the resulting force on a test particle with charge e

$$E_r = \frac{\lambda}{2\pi\epsilon_0}\frac{r}{a^2}$$

$$B_\phi = \frac{\lambda\beta}{2\pi\varepsilon_0 c}\frac{r}{a^2}$$



Lorentz force for the geometry studied

$$F_r = e\left(E_r - v_s B_\phi\right)$$

$$F_r = \frac{e\lambda}{2\pi\varepsilon_0}\left(1-\beta^2\right)\frac{r}{a^2} = \frac{e\lambda}{2\pi\varepsilon_0}\frac{1}{\gamma^2}\frac{r}{a^2}$$

Electric and magnetic components have opposite signs and scale between them with β^2 → there is perfect compensation when β=1

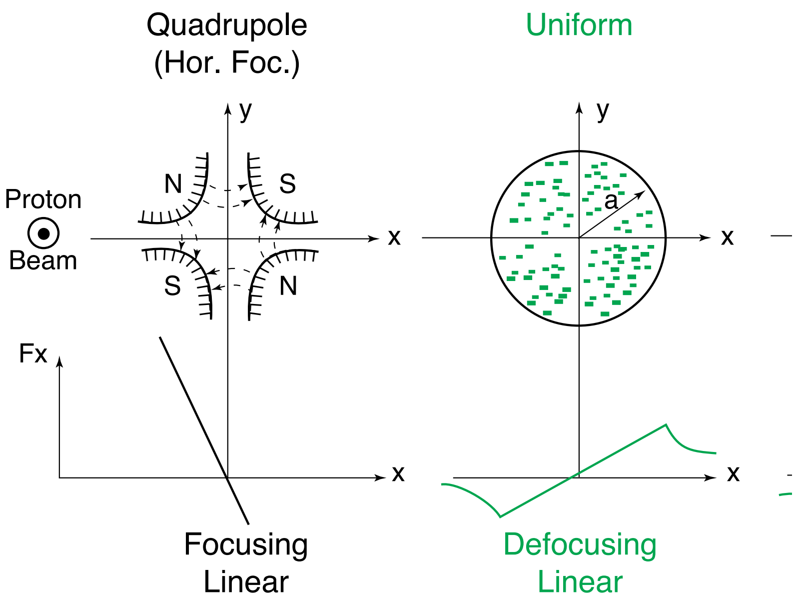
writing the force in x and y:

$$F_x = \frac{e\lambda}{2\pi\varepsilon_0\gamma^2 a^2}x$$

$$F_y = \frac{e\lambda}{2\pi\varepsilon_0\gamma^2 a^2}y$$

Coasting beam with uniform charge density

- Assume a coasting beam of circular cross section with radius *a* and uniform charge density $\eta=\lambda/\pi a^2$ [Cb/m³] moving at constant velocity $v=\beta c$
- In this case the direct space charge force is linear in x and y



$$F_x = \frac{e\lambda}{2\pi\varepsilon_0\gamma^2 a^2}x$$

$$F_y = \frac{e\lambda}{2\pi\varepsilon_0\gamma^2 a^2}y$$

direct space charge is like a defocusing quadrupole…

however, direct space charge is always defocusing in both planes, while quadrupole is focusing in one and defocusing in the other plane

Coasting beam with uniform charge density

- The direct space charge force for a beam with uniform charge distribution is linear in x and y → results in the **direct space charge tune shift**
  - We derive it here for the vertical plane …
  - Express *y”* in terms of F_y using Newton’s 2nd law

$$F_y = \frac{e\lambda}{2\pi\varepsilon_0\gamma^2 a^2}y$$

$$y'' = \frac{1}{\beta^2c^2}\frac{d^2y}{dt^2} = \frac{1}{\beta^2c^2}\frac{F_y}{m\gamma} = \frac{2r_0\lambda}{ea^2\beta^2\gamma^3}y$$

classical particle radius

$$r_0 = e^2/(4\pi\varepsilon_0 mc^2)$$

  - Generalize Hill’s equation including defocusing space charge term

$$\left.\begin{aligned} y'' + K_y(s)\,y = 0 \\ K_y^{SC}(s) = -\frac{2r_0\lambda}{ea^2(s)\beta^2\gamma^3}\end{aligned}\right\} \Longrightarrow \quad y'' + \left(K_y(s) - \frac{2r_0\lambda}{ea^2\beta^2\gamma^3}\right)y = 0$$

Generalized Hill’s equation

  - Calculate the tune shift treating space charge like a focusing error

$$\Delta Q_y = \frac{1}{4\pi}\oint K_y^{SC}(s)\beta_y(s)ds = -\frac{1}{4\pi}\oint\frac{2r_0\lambda\beta_y(s)}{ea^2(s)\beta^2\gamma^3}ds = -\frac{r_0R\lambda}{e\beta^2\gamma^3}\left\langle\frac{\beta_y(s)}{a^2(s)}\right\rangle$$

Coasting beam with uniform charge density

- After some reshuffling we notice that the **direct space charge tune shift**
  - is negative, because space charge transversely always defocuses
  - is proportional to the line density and thus to the number of particles in the beam
  - decreases with energy like $\beta^{-1}\gamma^{-2}$ (when expressed in terms of normalized emittance) and therefore vanishes in the ultrarelativistic limit
  - is does not depend on the local beta functions or beam sizes but is inversely proportional to the normalized emittance **(here the emittance includes all particles!)**

$$\left.\begin{aligned} \Delta Q_{x,y} &= -\frac{r_0 R\lambda}{e\beta^2\gamma^3}\left\langle\frac{\beta_{x,y}(s)}{a^2(s)}\right\rangle \\ a(s) &= \sqrt{\beta_{x,y}(s)\hat{\varepsilon}^n_{x,y}/\beta\gamma} \end{aligned}\right\}$$

$$\boxed{\Delta Q_{x,y} = -\frac{r_0 R\lambda}{e\beta\gamma^2\hat{\varepsilon}^n_{x,y}}}$$

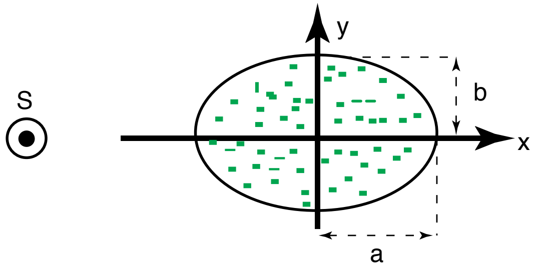
$$r_0 = e^2/(4\pi\varepsilon_0 mc^2) = \begin{cases} 1.54\cdot 10^{-18}\ \text{m (proton)} \\ 2.82\cdot 10^{-15}\ \text{m (electron)} \end{cases}$$

Coasting beam with uniform charge density

- Up to now we have assumed a circular cross section of the beam
  - This is quite unrealistic (think about the variation of the horizontal and vertical beta functions along a FODO cell)
- Beam with elliptical cross section with dimensions *a* and *b*
  - The space charge force is still linear and can be computed analytically
  - The tune shift is computed following the same steps as for the round beam
  - If beam size is smaller in one plane → larger space charge tune shift in that plane

$$\vec{E} = \frac{\lambda}{\pi\varepsilon_0(a+b)}\left(\frac{x}{a}, \frac{y}{b}, 0\right)$$

$$\vec{B} = \frac{\mu_0\lambda\beta c}{\pi(a+b)}\left(-\frac{y}{b}, \frac{x}{a}, 0\right)$$



$$\Rightarrow \quad K_x^{SC}(s) = -\frac{4r_0\lambda}{ea(a+b)\beta^2\gamma^3}$$

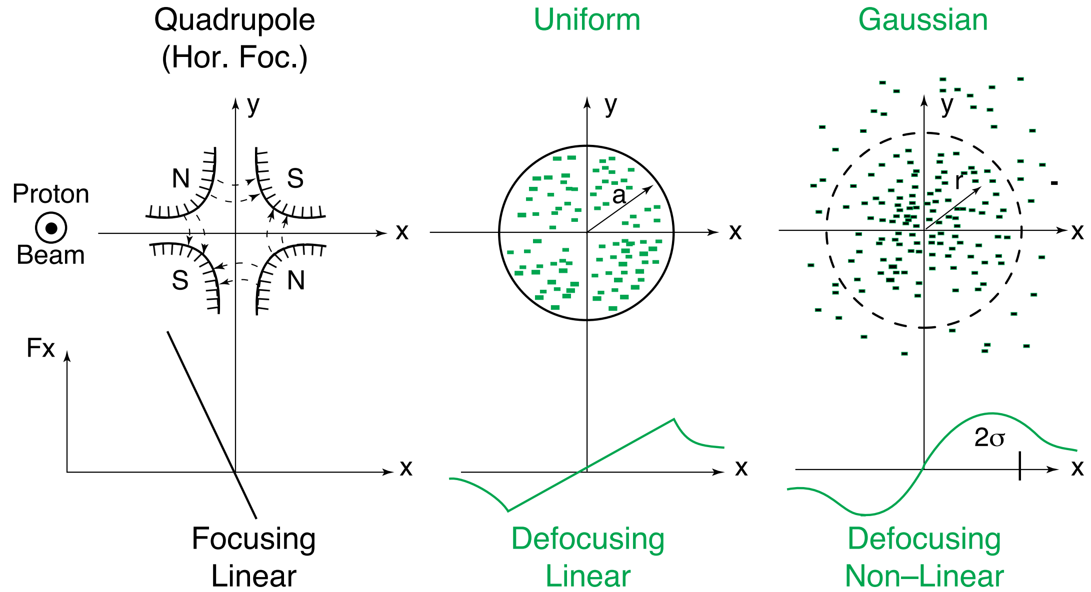
$$K_y^{SC}(s) = -\frac{4r_0\lambda}{eb(a+b)\beta^2\gamma^3}$$

$$\Rightarrow \quad \Delta Q_x = -\frac{r_0\lambda}{e\pi\beta\gamma^2\hat{\varepsilon}_x^n}\oint \frac{a}{a+b}ds$$

$$\Delta Q_y = -\frac{r_0\lambda}{e\pi\beta\gamma^2\hat{\varepsilon}_y^n}\oint \frac{b}{a+b}ds$$

Coasting beam with Gaussian distribution

- For the (quite realistic) case of a beam with a transverse bi-Gaussian distribution the space charge force becomes nonlinear
  - Beam field over the beam cross section is nonlinear
  - Particles will see different tune shifts according to their betatron amplitudes. This causes a tune spread rather than a tune shift!
  - We can calculate the maximum tune shift, i.e. the tune shift in the beam center …



Coasting beam with Gaussian distribution

- We consider first the case of a **circular beam** with rms beam size σ

$$\eta(r) = \frac{\lambda}{2\pi\sigma^2} e^{-r^2/2\sigma^2} \qquad \text{with rms beam size} \quad \sigma = \sqrt{\beta_{x,y}\varepsilon^n_{x,y}/\beta\gamma}$$

  - The following fields satisfy Maxwell's equations
  - We obtain the radial Lorentz force and linearize it for small *r* to calculate the tune shift for particles around the beam center

$$E_r(r) = \frac{\lambda}{2\pi\varepsilon_0}\frac{1}{r}\left(1 - e^{-r^2/2\sigma^2}\right)$$

$$B_\phi(r) = \frac{\lambda\beta}{2\pi\varepsilon_0 c}\frac{1}{r}\left(1 - e^{-r^2/2\sigma^2}\right)$$

$$F_r(r) = \frac{e\lambda}{2\pi\varepsilon_0\gamma^2}\frac{1}{r}\left(1 - e^{-r^2/2\sigma^2}\right)$$

$$F_r(r) = \frac{e\lambda}{2\pi\varepsilon_0\gamma^2}\frac{1}{r}\left(1 - 1 + \frac{r^2}{2\sigma^2} - \ldots\right) \approx \frac{e\lambda}{2\pi\varepsilon_0\gamma^2}\frac{r}{2\sigma^2}$$

maximum tune shift for a Gaussian beam distribution

$$\Longrightarrow \quad \Delta Q_{x,y} = -\frac{r_0\lambda}{2\pi e\beta^2\gamma^3}\oint\frac{\beta_{x,y}}{2\sigma^2}ds = -\frac{r_0 R\lambda}{e\beta\gamma^2}\frac{1}{2\varepsilon^n_{x,y}}$$

Coasting beam with Gaussian distribution

- We generalize the **direct space charge tune shift for particles in the center of a Gaussian beam** to the case of elliptical cross sections
  - We use the general expression of the beam size
  - In the vertical plane there is no dispersion and thus the formula can be expressed in terms of normalized emittance → we observe again the dependence on $\beta^{-1}\gamma^{-2}$

$$\sigma_x(s) = \sqrt{\beta_x(s)\varepsilon_x^n/\beta\gamma + D_x^2(s)\delta_{\text{rms}}^2}$$

$$\sigma_y(s) = \sqrt{\beta_y(s)\varepsilon_y^n/\beta\gamma}$$

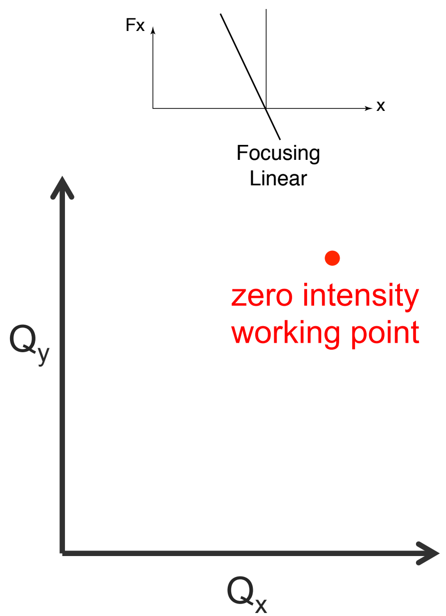
direct space charge tune shift for particles in the center of a Gaussian beam

$$\Delta Q_x = -\frac{r_0\lambda}{2\pi e\beta^2\gamma^3}\oint\frac{\beta_x(s)}{\sigma_x(s)\left[\sigma_x(s)+\sigma_y(s)\right]}ds$$

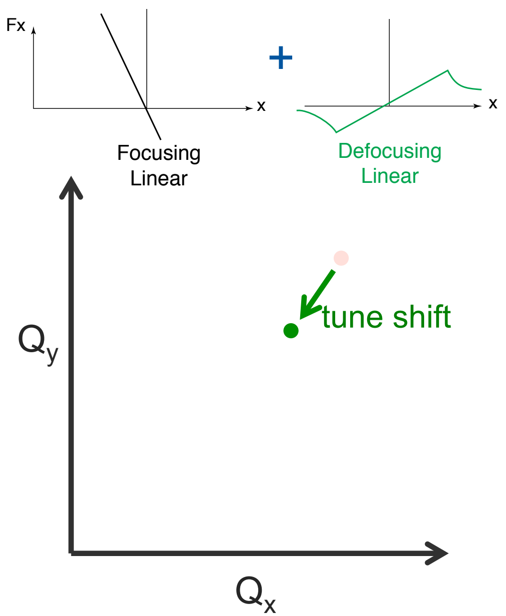
$$\Delta Q_y = -\frac{r_0\lambda}{2\pi e\beta^2\gamma^3}\oint\frac{\beta_y(s)}{\sigma_y(s)\left[\sigma_x(s)+\sigma_y(s)\right]}ds = -\frac{r_0R\lambda}{e\beta\gamma^2\varepsilon_y^n}\left\langle\frac{\sigma_y}{\sigma_x+\sigma_y}\right\rangle$$

Coasting beam – uniform vs. Gaussian

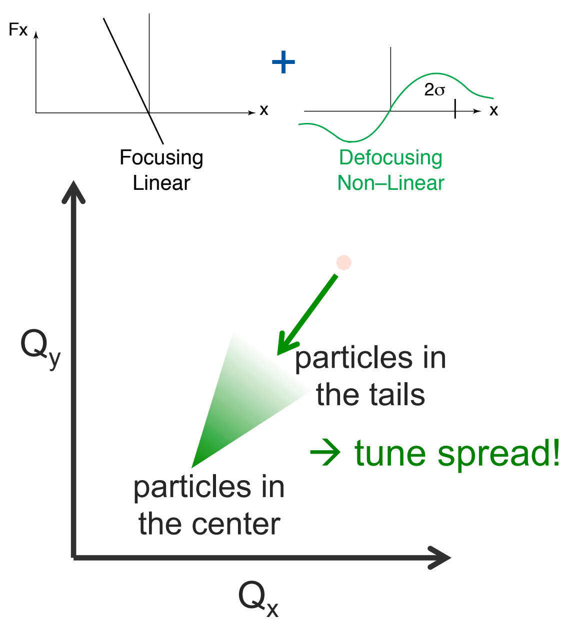
without space charge



with space charge
uniform distribution



with space charge
Gaussian distribution

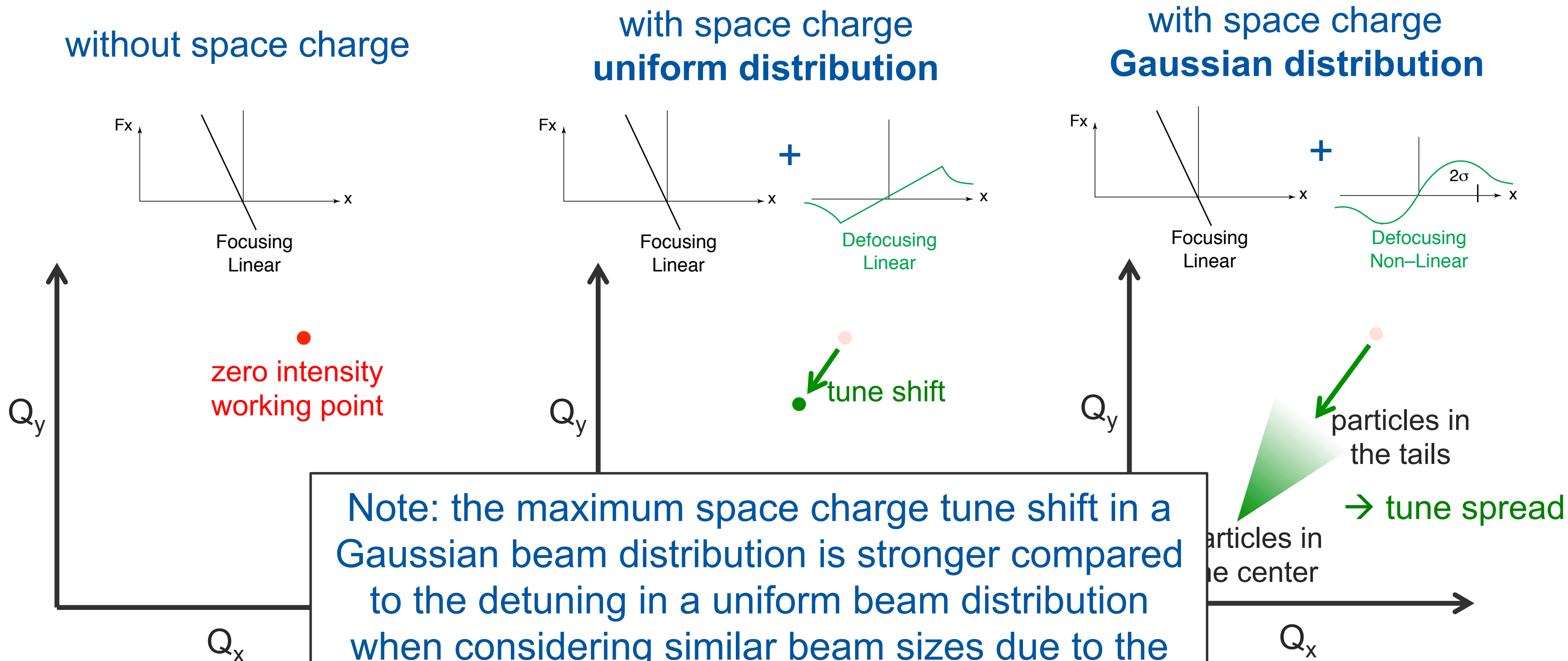


- All particles have the tunes Q_x and Q_y determined by the machine quadrupoles

- All particles have the tunes Q_x and Q_y determined by the machine quadrupoles and the linear defocusing from space charge

- Particles have a different tunes, since the space charge defocusing depends on the particles' amplitude
- The tune shift is largest for particles in the beam center

Coasting beam – uniform vs. Gaussian



Note: the maximum space charge tune shift in a Gaussian beam distribution is stronger compared to the detuning in a uniform beam distribution when considering similar beam sizes due to the higher particle density in the beam core!

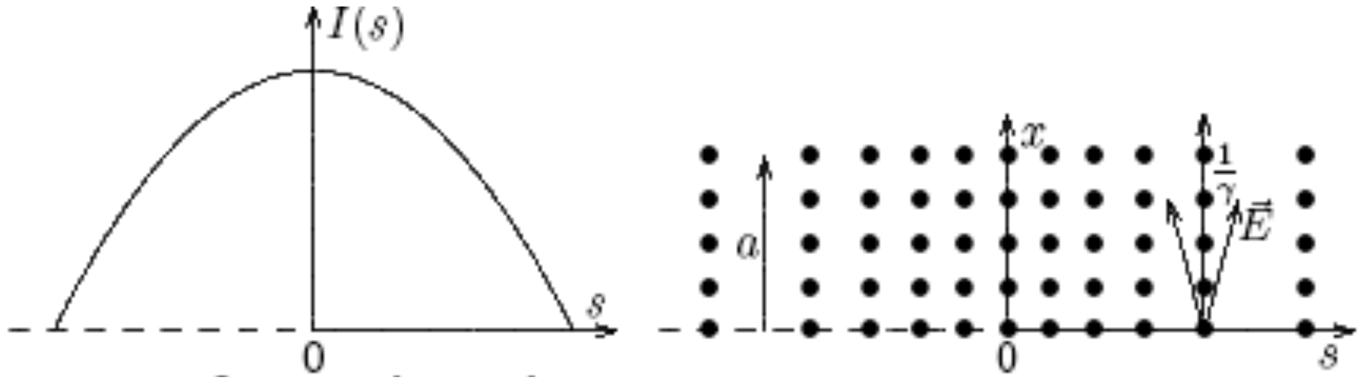
- All particles have … Q_x and Q_y determined by the machine quadrupoles

Q_x and Q_y determined by the machine quadrupoles and the linear defocusing from space charge

- …es have a different tunes, since the space charge defocusing depends on the particles' amplitude
- The tune shift is largest for particles in the beam center

Bunched beams

- If the beam is bunched and has an s-dependent line density
  - The relativistic field has an opening angle of $1/\gamma$
  - We can look at the bunched beam as locally continuous if the density $\lambda(s)$ changes smoothly (i.e. does not change much over $\Delta s=a/\gamma$ which means if $\gamma >> a/\Delta s$)
  - In this case, we can still use the formulas derived for coasting beams but use the s (or in fact z) dependent line density → the tune shift will depend on the position of the particle along the bunch
  - This translates into tune modulation with twice the synchrotron period, because particles execute longitudinal oscillations and sample different line densities



Longitudinal bunch profiles

- Uniform line density (i.e. coasting beam)

$$\lambda(z) = \lambda = \frac{Nq}{2\pi R}$$

N … total number of particles

- Bunched beams

$$N = n_b N_b$$

n_b … number of bunches

N_b … intensity per bunch

  - Gaussian line density

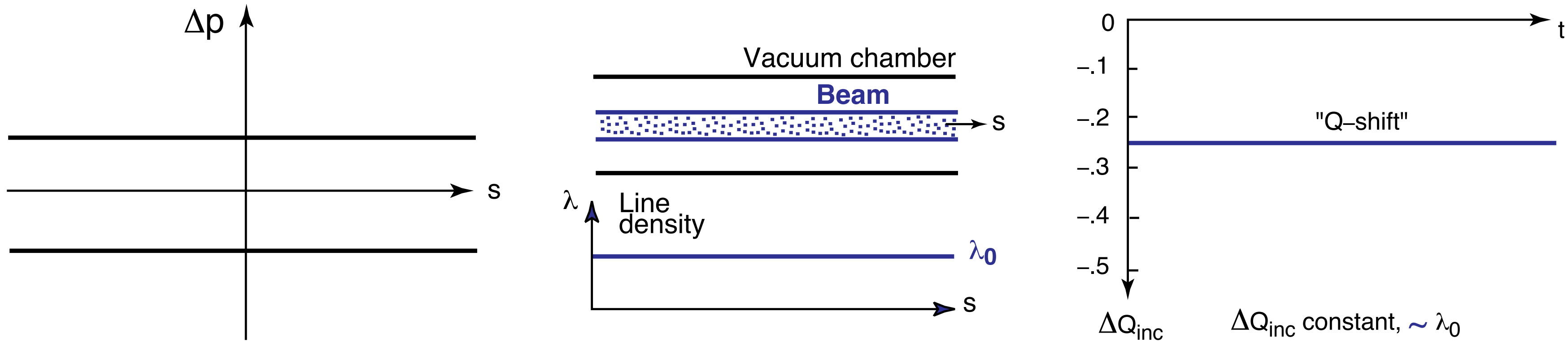
$$\lambda(z) = \frac{N_b q}{\sqrt{2\pi}\sigma_z} e^{-\frac{z^2}{2\sigma_z^2}} \longrightarrow \hat{\lambda} = \lambda_z(0) = \frac{N_b q}{\sqrt{2\pi}\sigma_z}$$

  - Parabolic line density for a single bunch

$$\lambda(z) = \begin{cases} \frac{3N_b q}{4\hat{z}^3}\left(\hat{z}^2 - z^2\right) & \text{if} \quad |z| \leq \hat{z} \\ 0 & \text{if} \quad |z| > \hat{z} \end{cases} \longrightarrow \hat{\lambda} = \lambda_z(0) = \frac{3N_b q}{4\hat{z}}$$

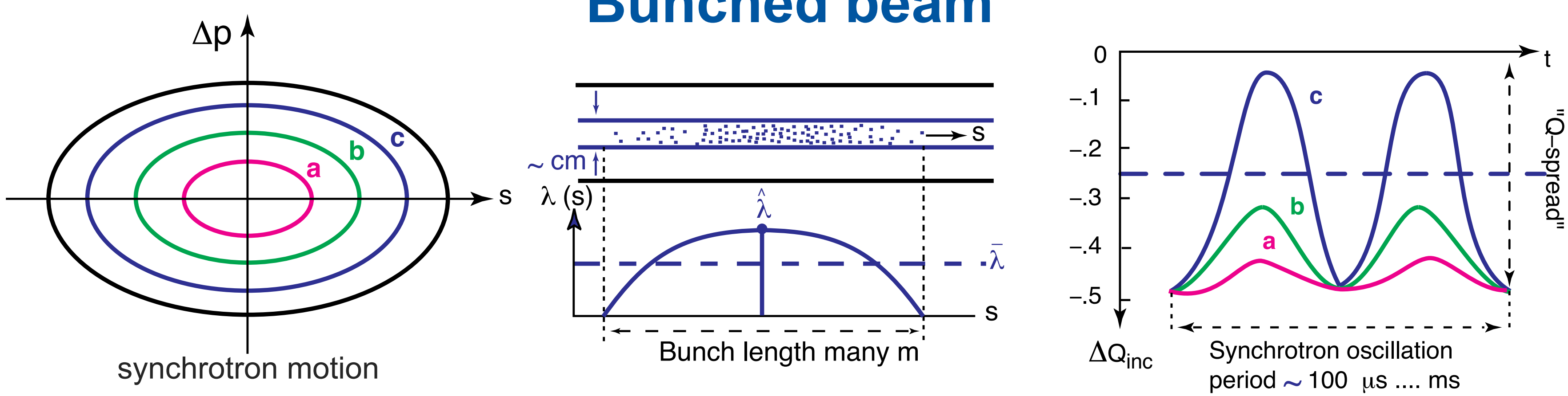
Coasting vs. bunched – uniform distribution

Coasting beam



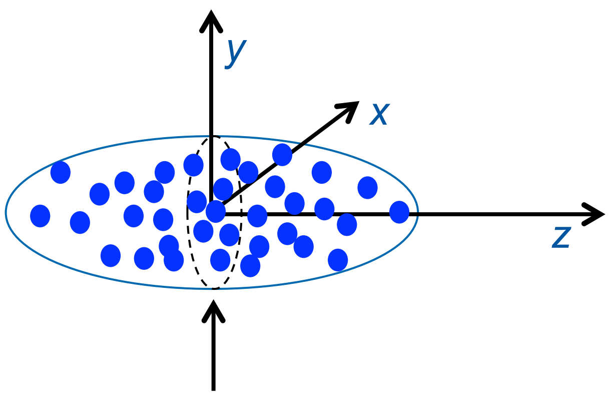
assuming that chromaticity is corrected …

Bunched beam

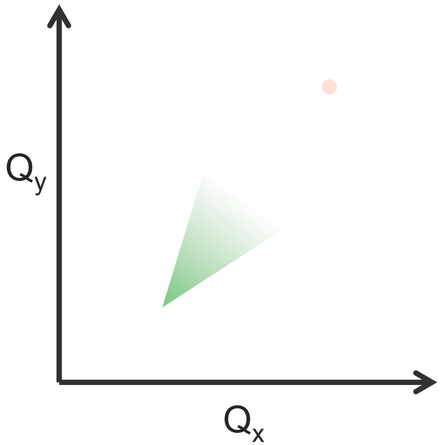


Coasting vs. bunched – Gaussian

- In beams with Gaussian transverse distribution we observed already for coasting beams with constant line density a tune spread due to the nonlinear force and the resulting dependence on the transvers particle amplitude
- In case a Gaussian beam is also bunched, an additional tune spread is induced by the variation of the line density

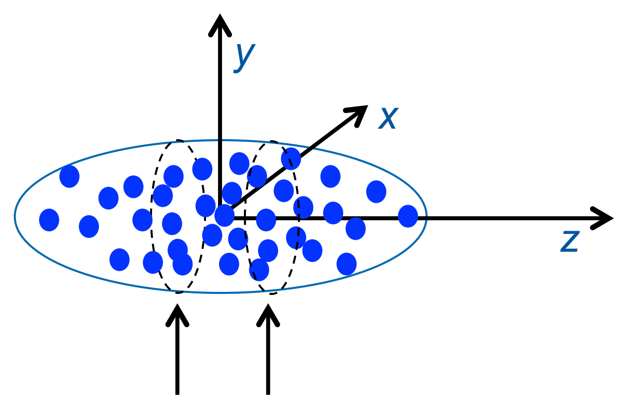


Particles close to the peak line density (often in the bunch center) will have the largest tune spread

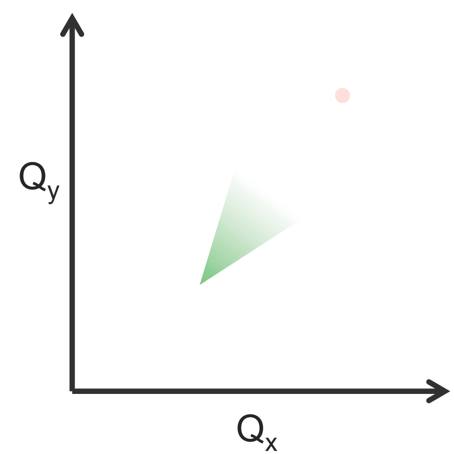


Coasting vs. bunched – Gaussian

- In beams with Gaussian transverse distribution we observed already for coasting beams with constant line density a tune spread due to the nonlinear force and the resulting dependence on the transvers particle amplitude
- In case a Gaussian beam is also bunched, an additional tune spread is induced by the variation of the line density

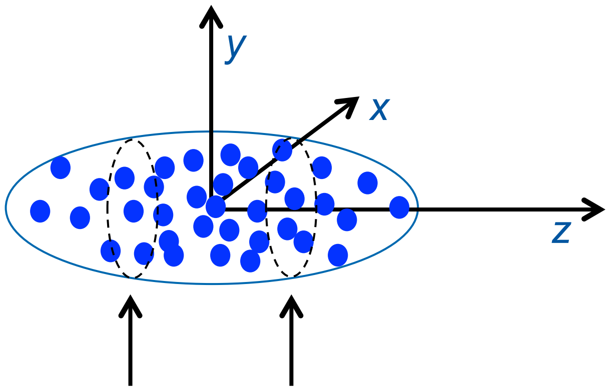


The tune spread is reduced for particles further away from the peak line density

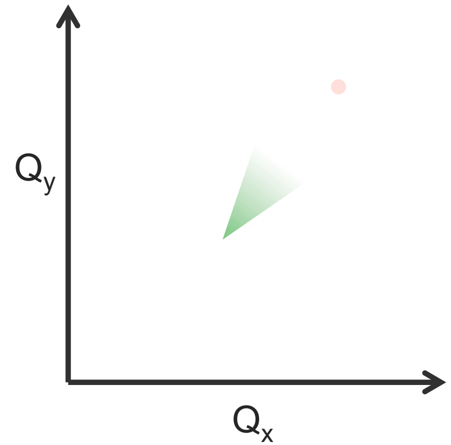


Coasting vs. bunched – Gaussian

- In beams with Gaussian transverse distribution we observed already for coasting beams with constant line density a tune spread due to the nonlinear force and the resulting dependence on the transvers particle amplitude
- In case a Gaussian beam is also bunched, an additional tune spread is induced by the variation of the line density

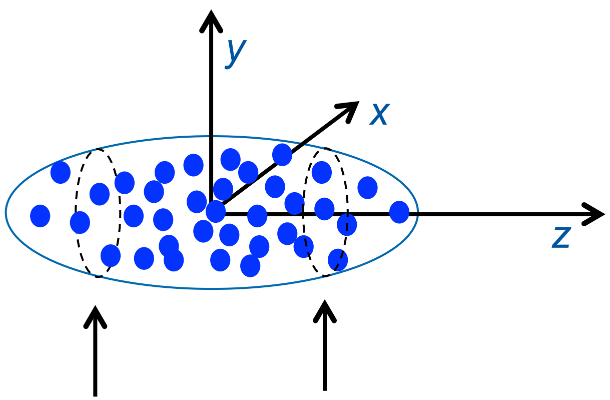


The tune spread is reduced for particles further away from the peak line density

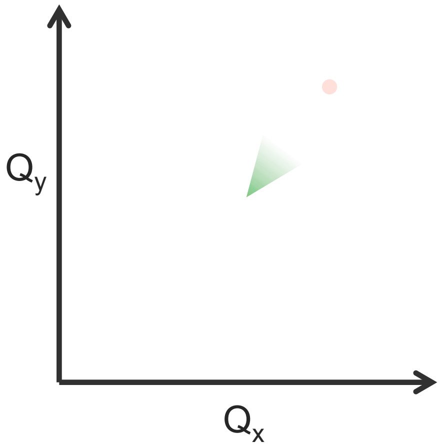


Coasting vs. bunched – Gaussian

- In beams with Gaussian transverse distribution we observed already for coasting beams with constant line density a tune spread due to the nonlinear force and the resulting dependence on the transvers particle amplitude
- In case a Gaussian beam is also bunched, an additional tune spread is induced by the variation of the line density

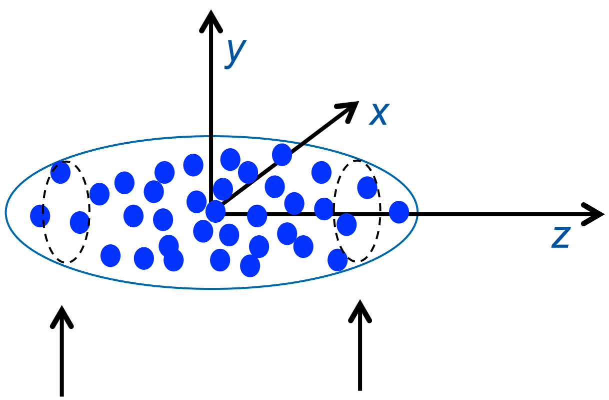


The tune spread is reduced for particles further away from the peak line density

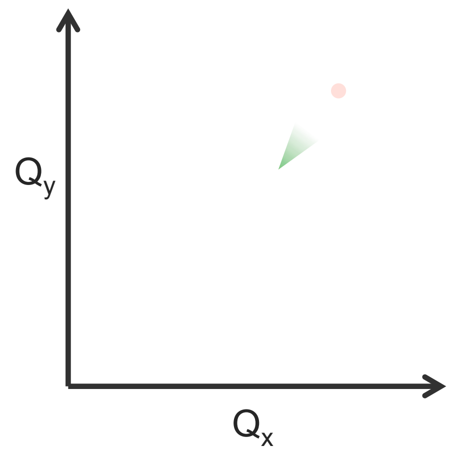


Coasting vs. bunched – Gaussian

- In beams with Gaussian transverse distribution we observed already for coasting beams with constant line density a tune spread due to the nonlinear force and the resulting dependence on the transvers particle amplitude
- In case a Gaussian beam is also bunched, an additional tune spread is induced by the variation of the line density

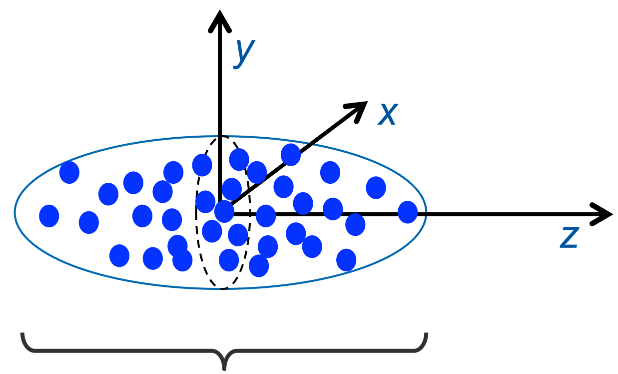


The tune spread is reduced for particles further away from the peak line density

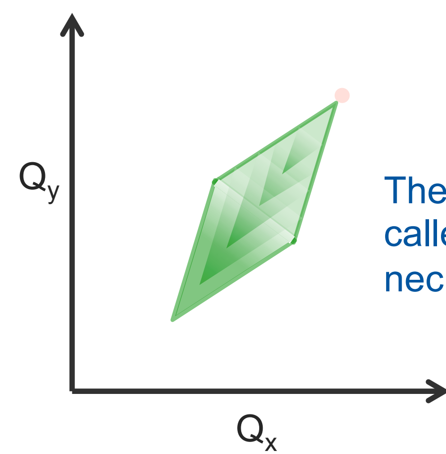


Coasting vs. bunched – Gaussian

- In beams with Gaussian transverse distribution we observed already for coasting beams with constant line density a tune spread due to the nonlinear force and the resulting dependence on the transvers particle amplitude
- In case a Gaussian beam is also bunched, an additional tune spread is induced by the variation of the line density
  - The longitudinal variation of the transverse space-charge force due to the line density fills the gap until the zero-intensity working point
  - The tune of individual particles is modulated by twice the synchrotron period



For bunched beams the tune spread is as big as the maximum tune shift!

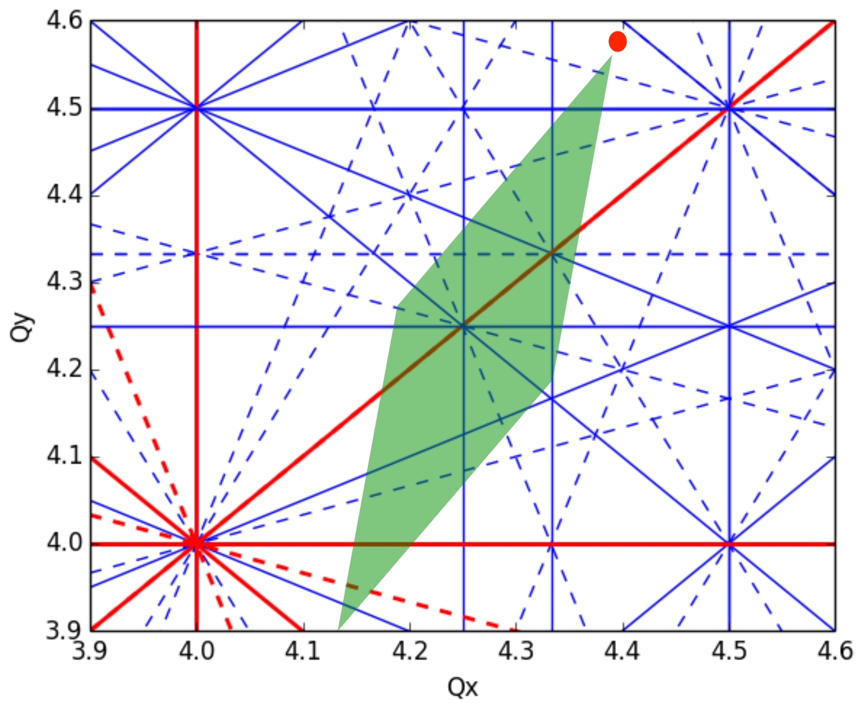


The tune footprint is also called the space charge necktie due to its shape

Brightness limitation due to space charge

- A space charge tune spread beyond 0.5 can barely be tolerated without excessive **emittance blow-up and/or particle loss due to resonances**
  - Dipole errors in the machine excite the integer resonances (Q=n)
  - Quadrupole errors excite the half integer resonances (Q=n+1/2)
  - Higher order resonances can be excited due to sextupoles and multipole errors

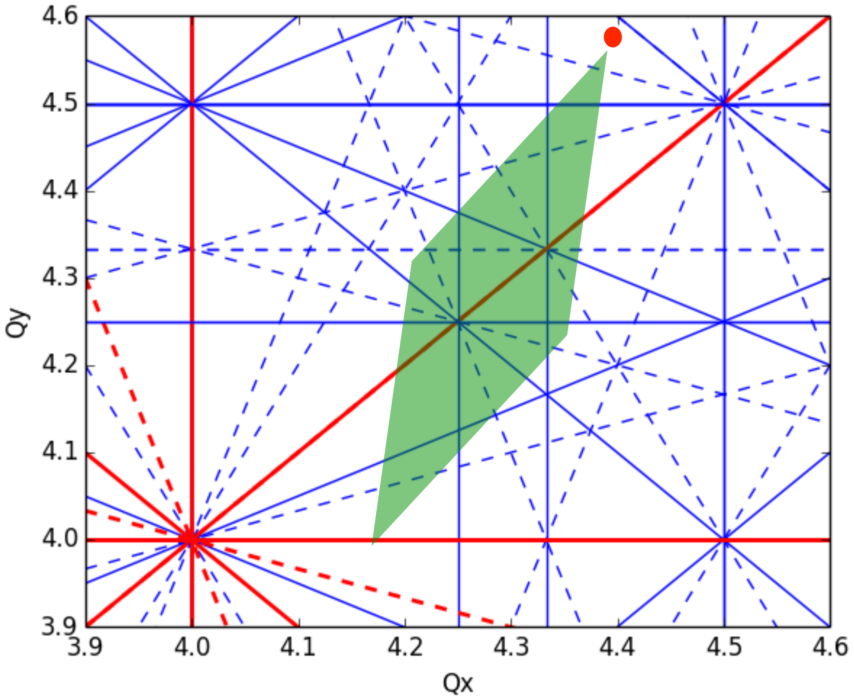
- Imagine that a beam with a tune spread of beyond 0.5 is injected



Brightness limitation due to space charge

- A space charge tune spread beyond 0.5 can barely be tolerated without excessive **emittance blow-up and/or particle loss due to resonances**
  - Dipole errors in the machine excite the integer resonances (Q=n)
  - Quadrupole errors excite the half integer resonances (Q=n+1/2)
  - Higher order resonances can be excited due to sextupoles and multipole errors

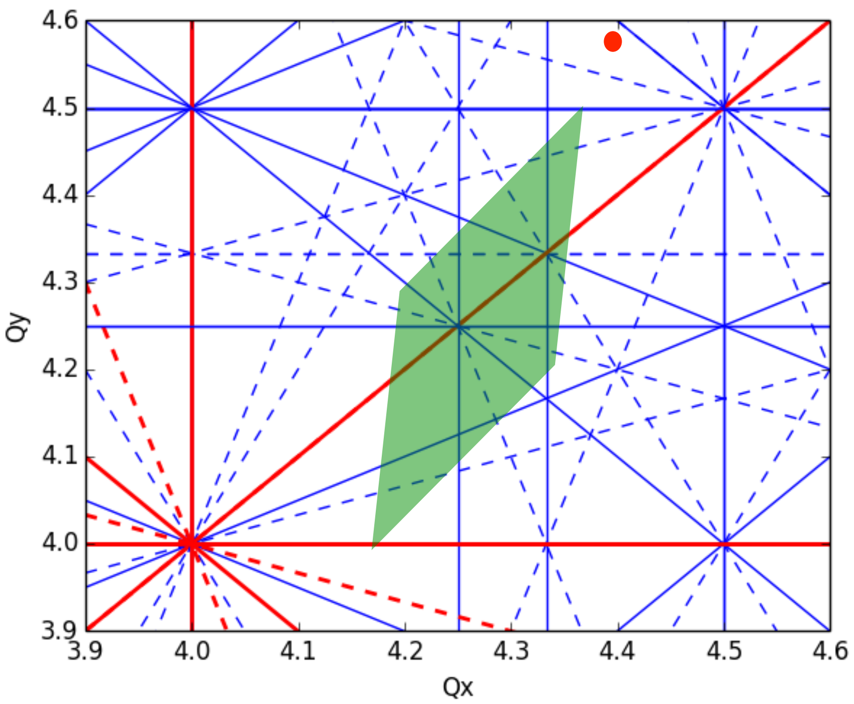
- Imagine that a beam with a tune spread of beyond 0.5 is injected
- Particles in the beam core will cross the integer resonance resulting in **emittance blow-up** and a reduction of the tune spread



Brightness limitation due to space charge

- A space charge tune spread beyond 0.5 can barely be tolerated without excessive **emittance blow-up and/or particle loss due to resonances**
  - Dipole errors in the machine excite the integer resonances (Q=n)
  - Quadrupole errors excite the half integer resonances (Q=n+1/2)
  - Higher order resonances can be excited due to sextupoles and multipole errors

- Imagine that a beam with a tune spread of beyond 0.5 is injected
- Particles in the beam core will cross the integer resonance resulting in **emittance blow-up** and a reduction of the tune spread
- Particles in the beam tails can be pushed onto the half integer resonance resulting in **losses due to aperture restrictions**

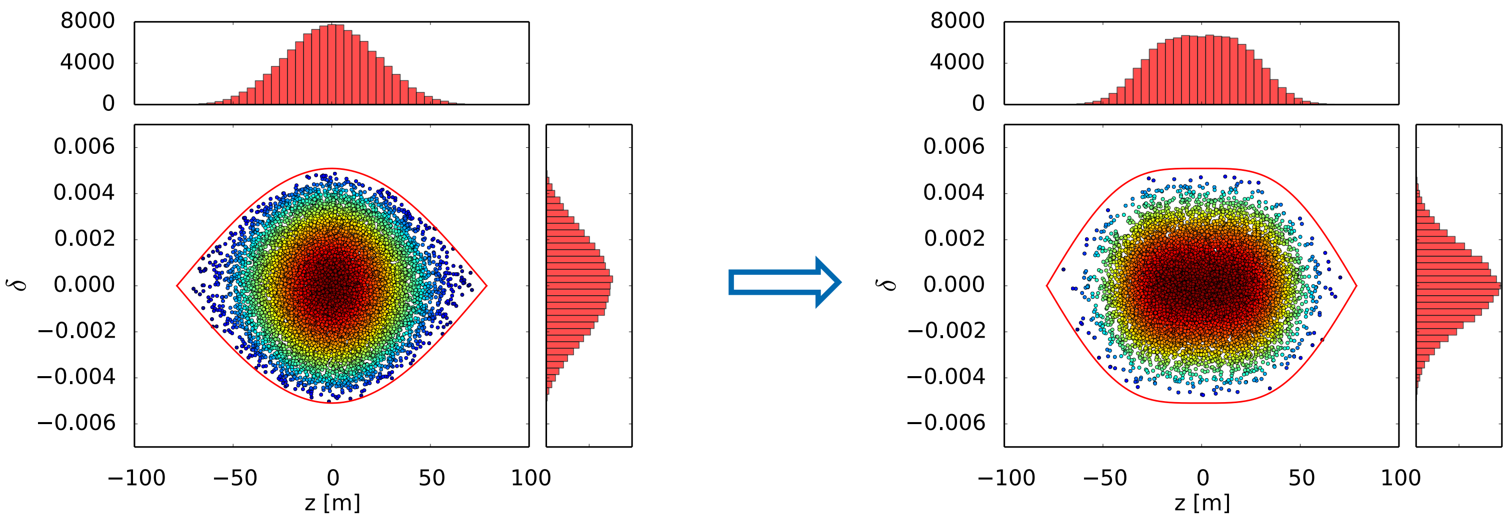


Mitigation techniques

Maximum tune shift (circular Gaussian)

$$\Delta\hat{Q}_{x,y} = -\frac{r_0 C \hat{\lambda}}{2\pi e \beta \gamma^2} \frac{1}{2\varepsilon^n_{x,y}}$$

- Decrease the peak line density by
  - maximizing the bunch length
  - flattening the bunch profile with a double harmonic RF system

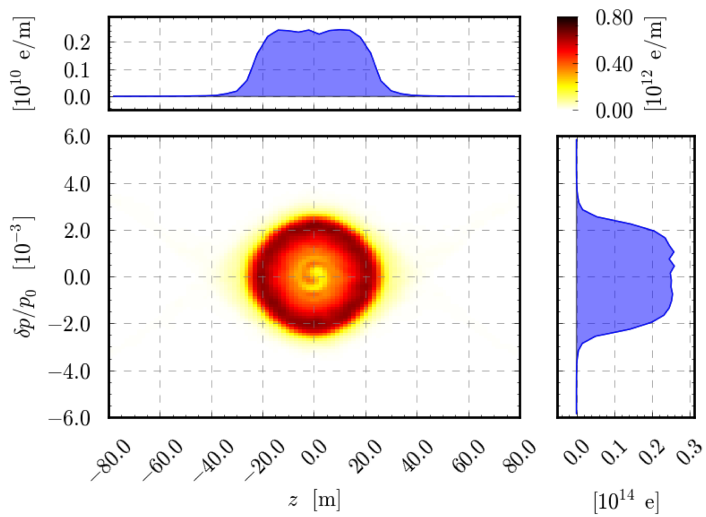


Mitigation techniques

Maximum tune shift (circular Gaussian)

$$\Delta\hat{Q}_{x,y} = -\frac{r_0 C \hat{\lambda}}{2\pi e \beta \gamma^2} \frac{1}{2\varepsilon^n_{x,y}}$$

- Decrease the peak line density by
  - maximizing the bunch length
  - flattening the bunch profile with a double harmonic RF system
  - using bunch distributions with small peak density (e.g. parabolic instead of Gaussian)
  - reducing the central density of the particle distribution (e.g. “hollow bunches”)



Mitigation techniques

Maximum tune shift (circular Gaussian)

$$\Delta\hat{Q}_{x,y} = -\frac{r_0 C \hat{\lambda}}{2\pi e \beta\gamma^2} \frac{1}{2\varepsilon_{x,y}^n}$$

- Decrease the peak line density by
  - maximizing the bunch length
  - flattening the bunch profile with a double harmonic RF system
  - using bunch distributions with small peak density (e.g. parabolic instead of Gaussian)
  - reducing the central density of the particle distribution (e.g. "hollow bunches")
- Increase the beam energy by
  - accelerating the beam as quickly as possible
  - increasing the injection energy (usually requires an upgrade of the pre-injector)

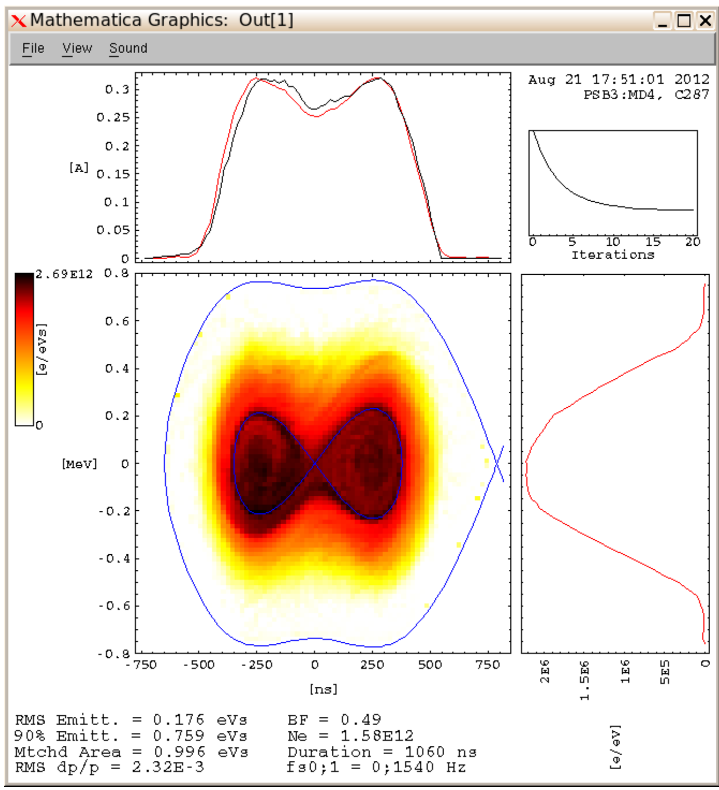
Mitigation techniques

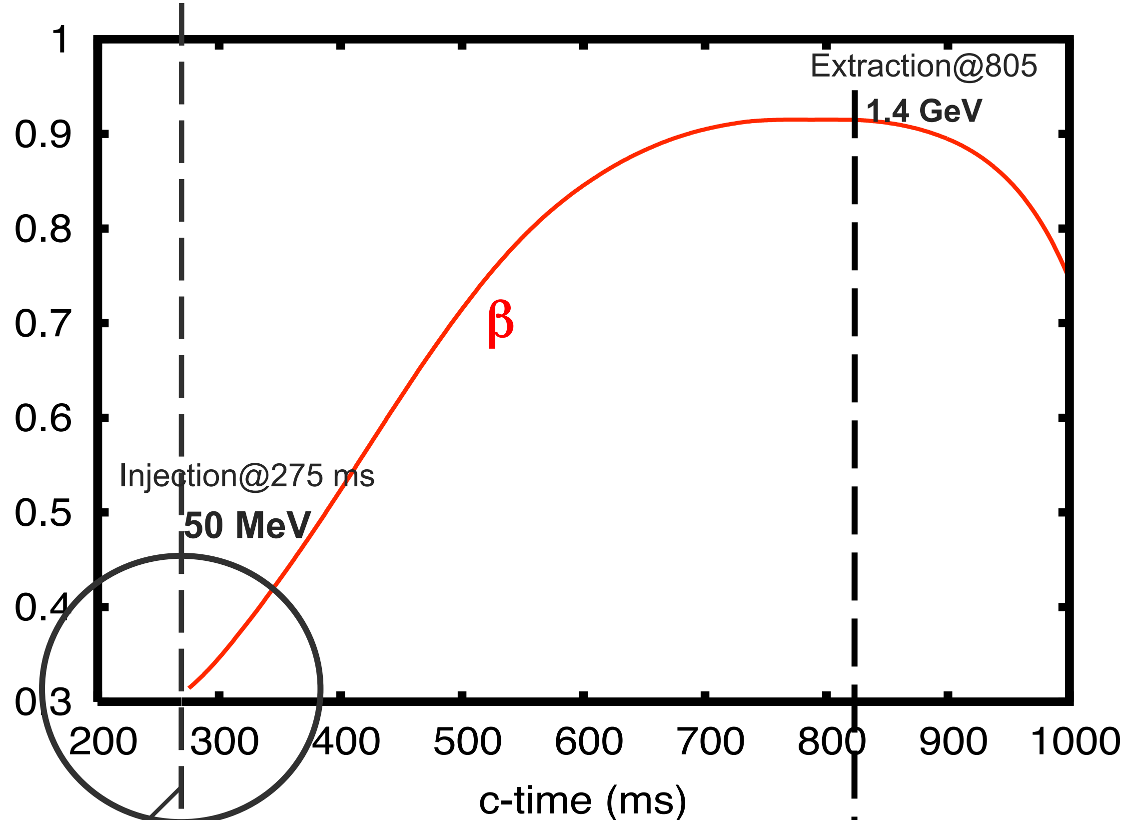
Maximum tune shift (circular Gaussian)

$$\Delta \hat{Q}_{x,y} = -\frac{r_0 C \hat{\lambda}}{2\pi e \beta \gamma^2} \frac{1}{2\varepsilon^n_{x,y}}$$

- Decrease the peak line density by
  - maximizing the bunch length
  - flattening the bunch profile with a double harmonic RF system
  - using bunch distributions with small peak density (e.g. parabolic instead of Gaussian)
  - reducing the central density of the particle distribution (e.g. “hollow bunches”)
- Increase the beam energy by
  - accelerating the beam as quickly as possible
  - increasing the injection energy (usually requires an upgrade of the pre-injector)
- Minimize the machine circumference
  - when designing/building a new accelerator since the space charge detuning is an integrated effect

Direct space charge in the CERN PSB

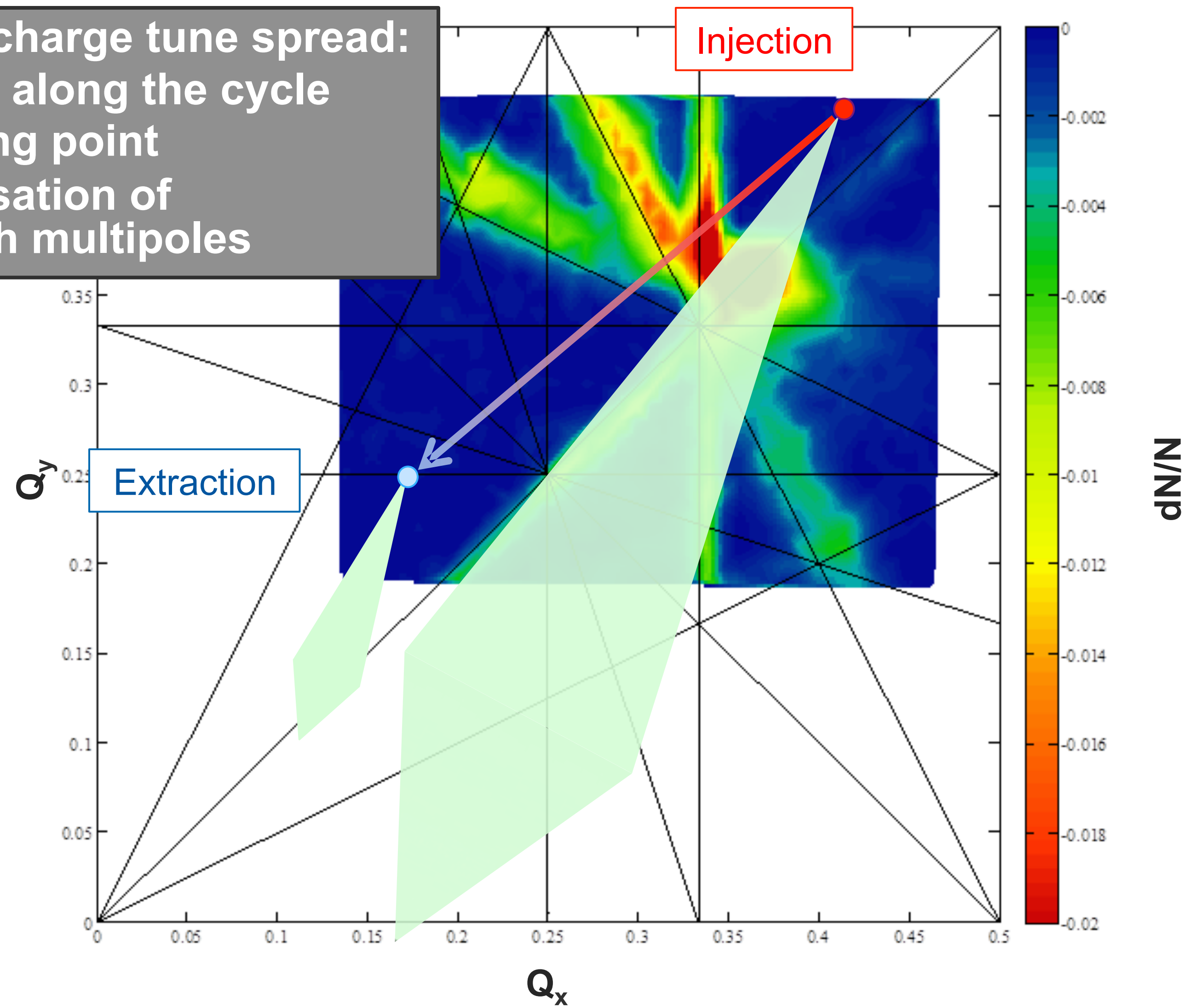




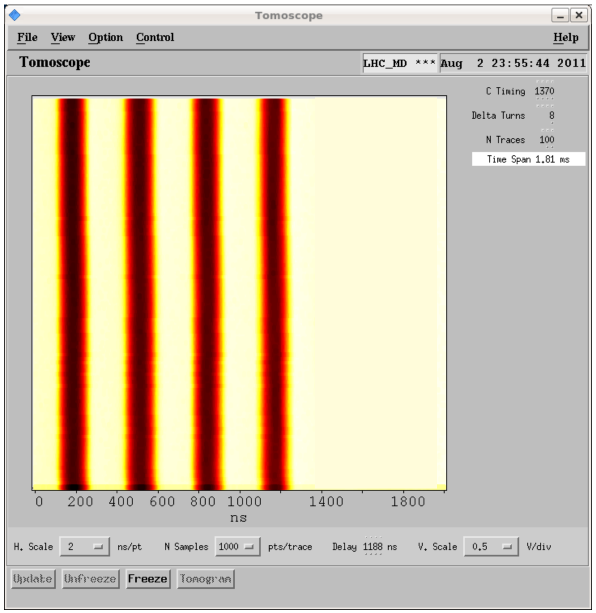
- The PSB accelerates bright beams from 50 MeV to 1.4 GeV over 530 ms
- Space charge important, especially in first part of the cycle – bunch is flattened through a second harmonic RF system

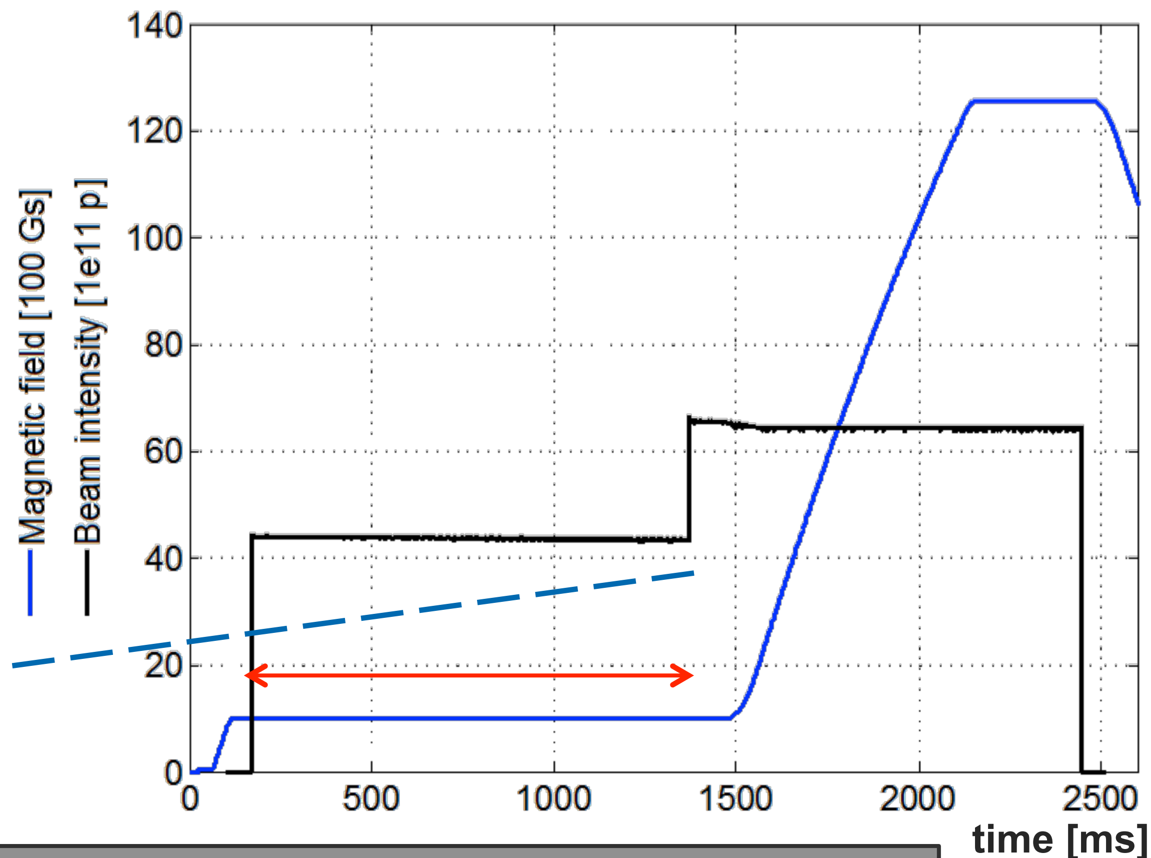
Direct space charge in the CERN PSB

To mitigate space charge tune spread:
- **Acceleration all along the cycle**
- **Dynamic working point**
- **Active compensation of resonances with multipoles**



Direct space charge in the CERN PS

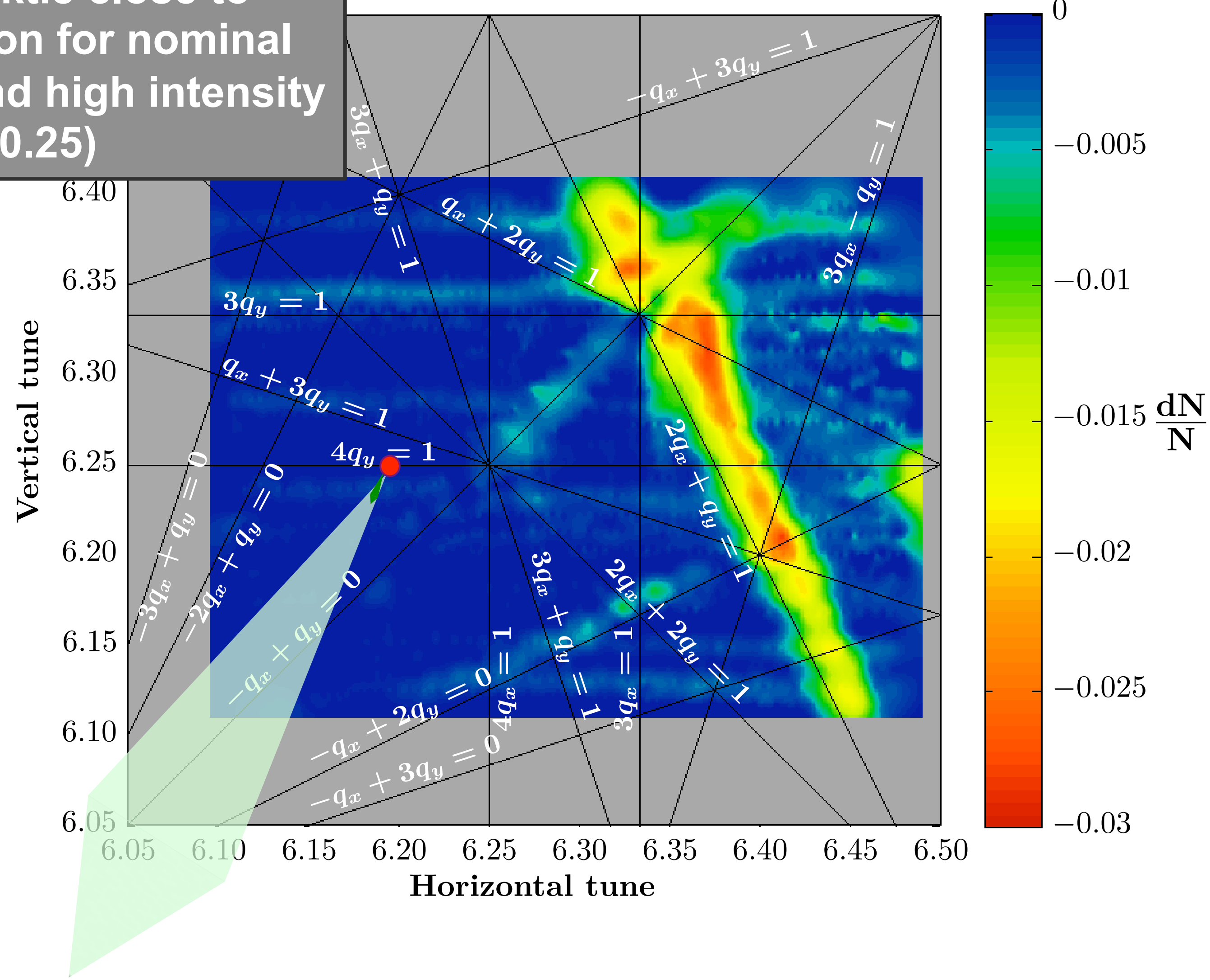




- First four bunches from PSB circulate in PS for 1.2 s and can suffer losses and emittance growth due to space charge
- The PS receives other two bunches from the PSB just before acceleration

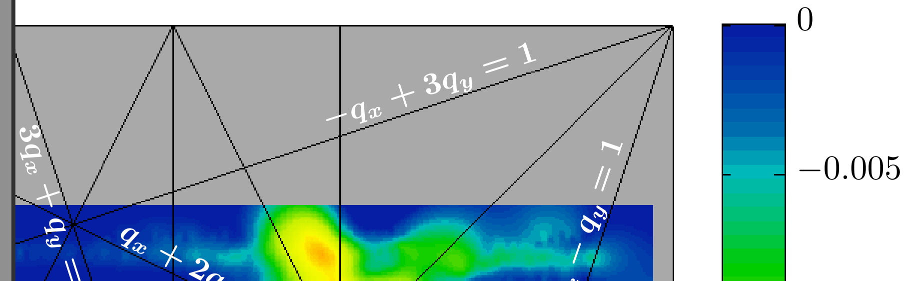
Direct space charge in the CERN PS

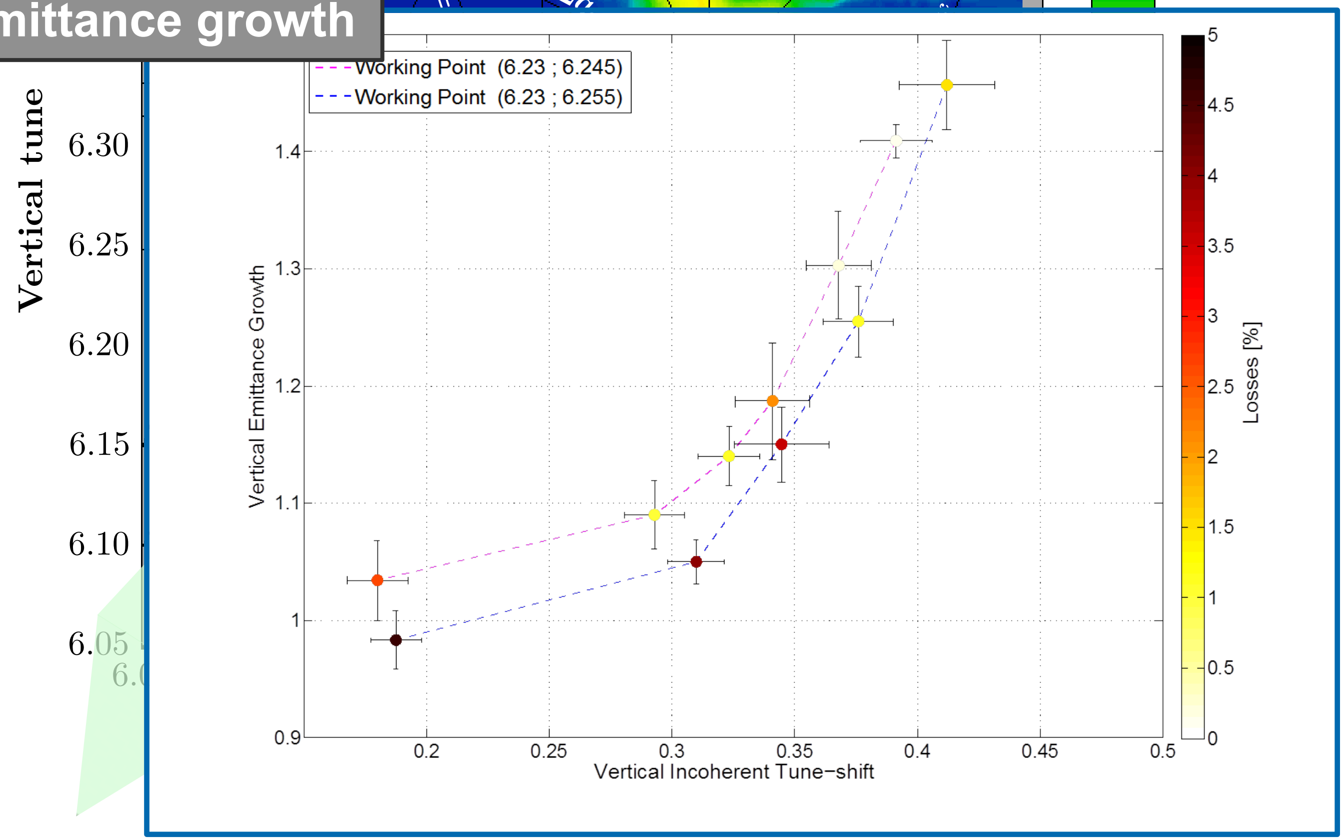
- Tune spread necktie close to integer at injection for nominal working point and high intensity LHC beam (ΔQ≈-0.25)



Direct space charge in the CERN PS

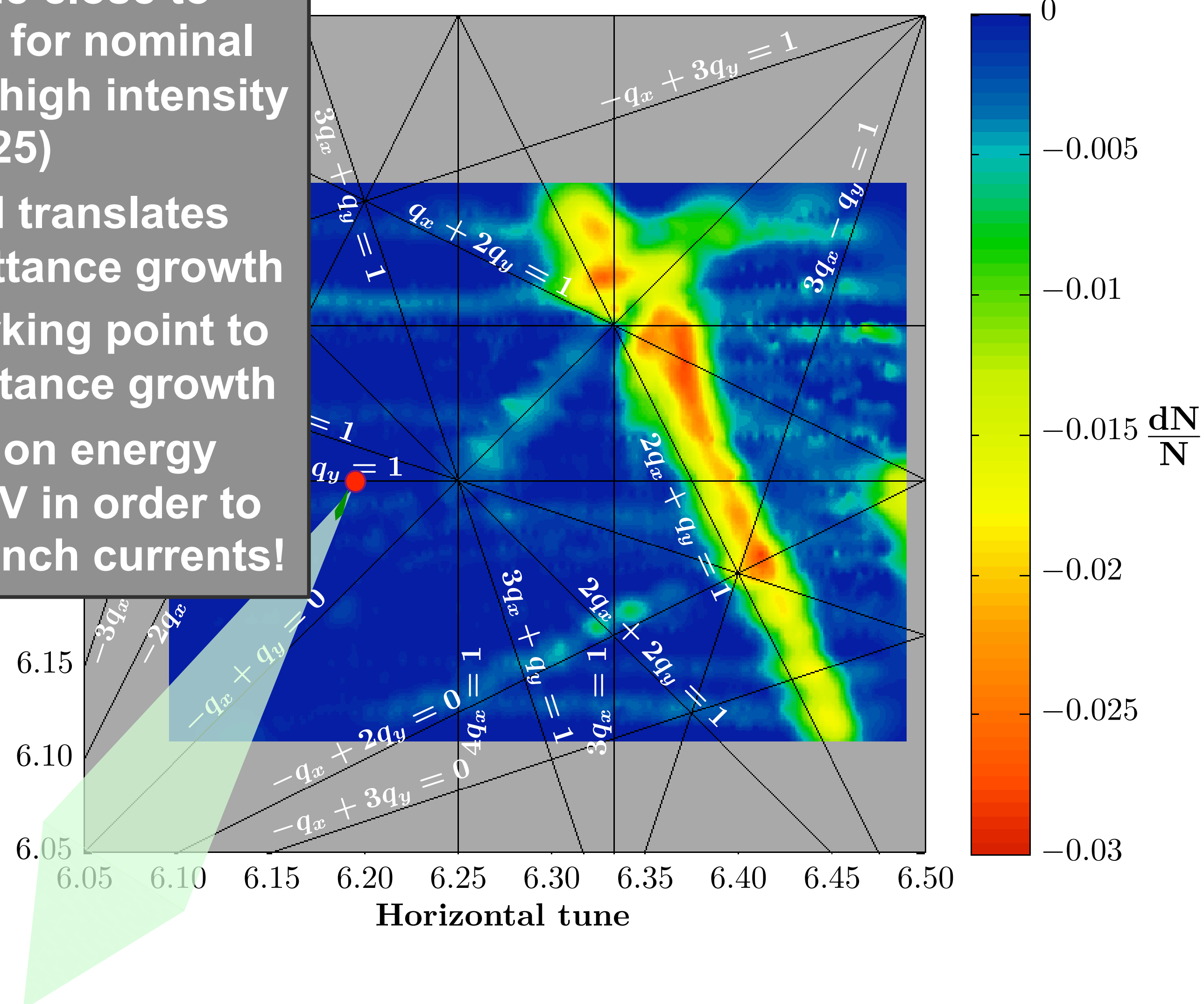
- Tune spread necktie close to integer at injection for nominal working point and high intensity LHC beam (ΔQ≈-0.25)
- Larger tune spread translates into important emittance growth





Direct space charge in the CERN PS

- Tune spread necktie close to integer at injection for nominal working point and high intensity LHC beam (ΔQ≈-0.25)
- Larger tune spread translates into important emittance growth
- Need to adjust working point to avoid losses / emittance growth
- Upgrade the injection energy into the PS to 2 GeV in order to allow for higher bunch currents!



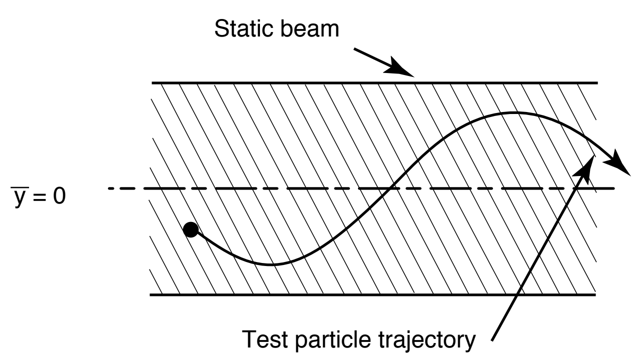
Outline

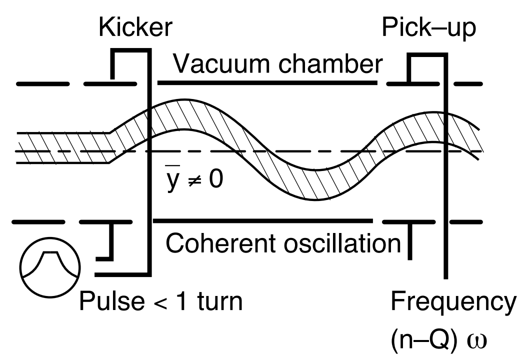
1. Introduction
   - Accelerator concepts
   - Coordinate systems and phase space
   - Transverse single particle dynamics
   - Longitudinal single particle dynamics
   - Collective effects

2. Space charge
   - Direct space charge (transverse)
   - Indirect space charge (transverse)
   - Longitudinal space charge

Incoherent vs. coherent

- Incoherent motion
  - The beam consists of many particles, each of which moves inside the beam with its individual betatron amplitude, phase, and even tune (under influence of direct space charge) – amplitude and phase are distributed at random over all particles
  - An outside observer (using a beam position monitor) does not see any of this random betatron motion – beam and source of the direct space-charge field do not move
- Coherent motion
  - A static beam given a transverse fast deflection starts to perform betatron oscillations as a whole – the source of direct space charge is now moving and individual particles continue their incoherent motion around the common coherent trajectory
  - In the accelerator environment, the coherently oscillating beam induces image charges/currents which are oscillating as well – this leads to a coherent tune shift





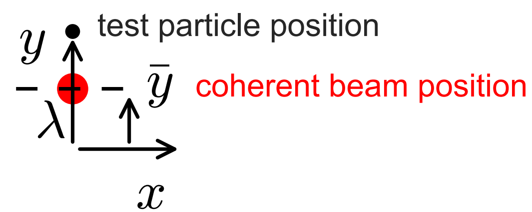
General tune shift expression

- We will now derive general expressions for the tune shift due to electromagnetic forces induced by the beam
  - From Newton's second law we have in general for the vertical plane

$$y'' = \frac{1}{\beta^2 c^2} \frac{d^2 y}{dt^2} = \frac{1}{\beta^2 c^2} \frac{F_y}{m\gamma}$$

  - We write the beam force at the position of test particle y as a function of the coherent beam position $\bar{y}$ and so the equation of motion (in smooth approximation) becomes

$$y'' + \left(\frac{Q_{y0}}{R}\right)^2 y = \frac{\langle F_{\text{beam}}(y, \bar{y}) \rangle}{\beta^2 c^2 m\gamma}$$



  - We are interested only in small oscillation amplitudes and so we can Taylor expand the force around zero to obtain

$$y'' + \left(\frac{Q_{y0}}{R}\right)^2 y = \frac{1}{\beta^2 c^2 m\gamma} \left(\frac{\partial \langle F_{\text{beam}} \rangle}{\partial y} \bigg|_{\bar{y}=0} y + \frac{\partial \langle F_{\text{beam}} \rangle}{\partial \bar{y}} \bigg|_{y=0} \bar{y} \right)$$

General tune shift expression - incoherent

$$y'' + \left(\frac{Q_{y0}}{R}\right)^2 y = \frac{1}{\beta^2 c^2 m\gamma}\left(\left.\frac{\partial\langle F_{\text{beam}}\rangle}{\partial y}\right|_{\bar{y}=0} y + \left.\frac{\partial\langle F_{\text{beam}}\rangle}{\partial \bar{y}}\right|_{y=0} \bar{y}\right)$$

- In case the induced tune shift is small we can write

$$Q_y^2 = Q_{y0}^2 + 2Q_{y0}\Delta Q_y + (\Delta Q_y)^2 \approx Q_{y0}^2 + 2Q_{y0}\Delta Q_y$$

- The incoherent tune shift of an individual test particle is obtained when the beam as a whole is centered
- In this case the second term in the general equation of motion becomes zero and we obtain the incoherent tune shift as

$$\left(Q_y^{\text{inc}}\right)^2 = Q_{y0}^2 + \frac{R^2}{\beta^2 c^2 m\gamma}\left.\frac{\partial\langle F_{\text{beam}}\rangle}{\partial y}\right|_{\bar{y}=0} \quad \Longrightarrow$$

$$\Delta Q_y^{\text{inc}} = \frac{R^2}{2Q_{y0}\beta^2 c^2 m\gamma}\left.\frac{\partial\langle F_{\text{beam}}\rangle}{\partial y}\right|_{\bar{y}=0} = \frac{R\langle\beta_y\rangle}{2\beta^2 c^2 m\gamma}\left.\frac{\partial\langle F_{\text{beam}}\rangle}{\partial y}\right|_{\bar{y}=0}$$

General tune shift expression – coherent

$$y'' + \left(\frac{Q_{y0}}{R}\right)^2 y = \frac{1}{\beta^2 c^2 m\gamma}\left(\left.\frac{\partial\langle F_{\text{beam}}\rangle}{\partial y}\right|_{\bar{y}=0} y + \left.\frac{\partial\langle F_{\text{beam}}\rangle}{\partial \bar{y}}\right|_{y=0} \bar{y}\right)$$

- For deriving the coherent tune shift we write the above equation for each individual particle and average over all particles to get the coherent beam motion

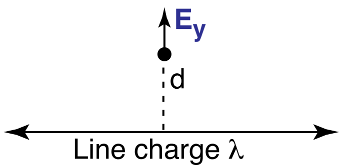
$$\bar{y}'' + \left(\frac{Q_{y0}}{R}\right)^2 \bar{y} = \frac{1}{\beta^2 c^2 m\gamma}\left(\left.\frac{\partial\langle F_{\text{beam}}\rangle}{\partial y}\right|_{\bar{y}=0} \bar{y} + \left.\frac{\partial\langle F_{\text{beam}}\rangle}{\partial \bar{y}}\right|_{y=0} \bar{y}\right)$$

- If the perturbation is small we obtain the coherent tune shift as

$$\begin{aligned}\Delta Q_y^{\text{coh}} &= \frac{R^2}{2Q_{y0}\beta^2 c^2 m\gamma}\left(\left.\frac{\partial\langle F_{\text{beam}}\rangle}{\partial y}\right|_{\bar{y}=0} + \left.\frac{\partial\langle F_{\text{beam}}\rangle}{\partial \bar{y}}\right|_{y=0}\right)\\ &= \frac{R\langle\beta_y\rangle}{2\beta^2 c^2 m\gamma}\left(\left.\frac{\partial\langle F_{\text{beam}}\rangle}{\partial y}\right|_{\bar{y}=0} + \left.\frac{\partial\langle F_{\text{beam}}\rangle}{\partial \bar{y}}\right|_{y=0}\right)\end{aligned}$$

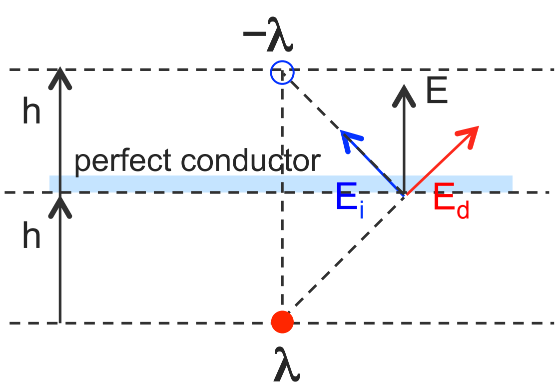
Electric image forces

- We consider the beam as line charge density λ with infinite length
  - The electric field at distance d is given by Gauss' law



$$E_y = \frac{\lambda}{2\pi\varepsilon_0}\frac{1}{d}$$

- The boundary condition at a perfect conducting surface requires that the electric field parallel to the plate vanishes
  - This boundary condition is satisfied by introducing an image charge

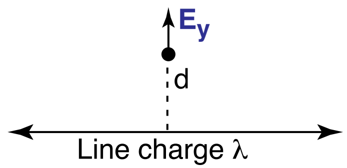


$$E_{||} = 0$$

Electric image forces

example: perfect conducting parallel plates

- We consider the beam as line charge density λ with infinite length
  - The electric field at distance d is given by Gauss' law

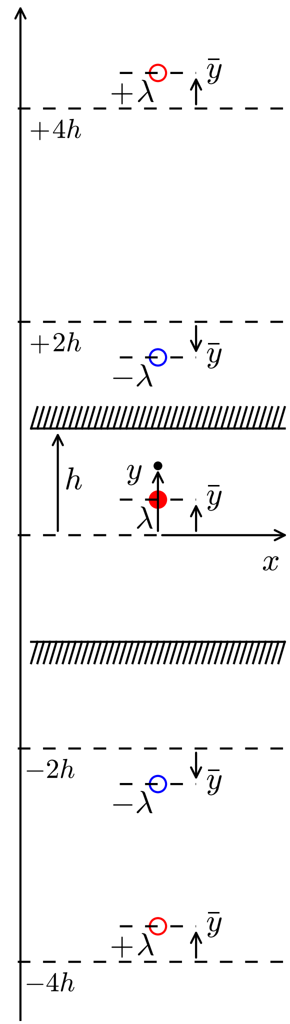


$$E_y = \frac{\lambda}{2\pi\varepsilon_0}\frac{1}{d}$$

- A flat vacuum chamber can be approximated by two perfect conducting parallel plates placed at vertical positions +h and –h
  - Let the beam be displaced vertically by y
  - The witness particle is at y

  $\bar{y}$, y << h

  - The boundary condition at the two perfect conducting plates $E_{||} = 0$ is satisfied by superposing an infinite number of image line charges with alternating signs as shown in the sketch



Electric image forces

example: perfect conducting parallel plates

$$E_y = \frac{\lambda}{2\pi\varepsilon_0}\left[+\frac{1}{2h-\bar{y}-y}-\frac{1}{2h+\bar{y}+y}+\frac{1}{6h-\bar{y}-y}-\frac{1}{6h+\bar{y}+y}+\dots\right.$$

$$\left.-\frac{1}{4h+\bar{y}-y}+\frac{1}{4h-\bar{y}+y}-\frac{1}{8h+\bar{y}-y}+\frac{1}{8h-\bar{y}+y}+\dots\right] =$$

group every two adjacent terms

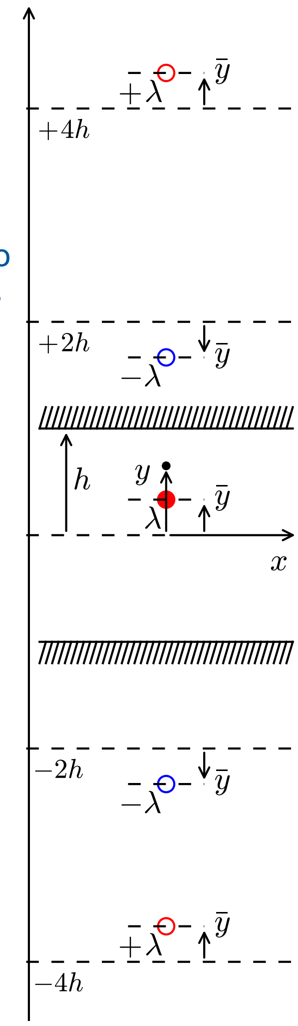
$$= \frac{\lambda}{2\pi\varepsilon_0}\left[+\frac{2(\bar{y}+y)}{(2h)^2-(\bar{y}+y)^2}+\frac{2(\bar{y}+y)}{(6h)^2-(\bar{y}+y)^2}+\dots\right.$$

$$\left.+\frac{2(\bar{y}-y)}{(4h)^2-(\bar{y}-y)^2}+\frac{2(\bar{y}-y)}{(8h)^2-(\bar{y}-y)^2}+\dots\right] =$$

Keep only linear terms

$$= \frac{\lambda}{\pi\varepsilon_0 h^2}\left[(\bar{y}+y)\left(\frac{1}{2^2}+\frac{1}{6^2}+\frac{1}{10^2}+\dots\right)+(\bar{y}-y)\left(\frac{1}{4^2}+\frac{1}{8^2}+\frac{1}{12^2}+\dots\right)\right] =$$

$$E_y = \frac{\lambda}{\pi\varepsilon_0 h^2}\left[(\bar{y}+y)\frac{\pi^2}{32}+(\bar{y}-y)\frac{\pi^2}{96}\right]$$



Electric image forces - incoherent

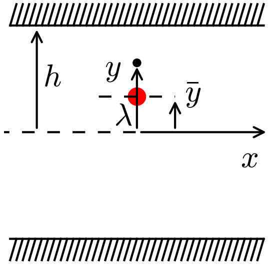
- In general the contribution from electric images to the *incoherent* beam force for a variety of geometries can be expressed in terms of **“Laslett coefficients”**

$$\left.\frac{\partial \langle F_{\text{beam}} \rangle}{\partial x}\right|_{\bar{x}=0} = \frac{e\lambda}{\pi\varepsilon_0}\frac{\varepsilon_1^x}{h^2}$$

$$\left.\frac{\partial \langle F_{\text{beam}} \rangle}{\partial y}\right|_{\bar{y}=0} = \frac{e\lambda}{\pi\varepsilon_0}\frac{\varepsilon_1^y}{h^2}$$

$$\varepsilon_1^x,\ \varepsilon_1^y \ \ldots \left\{ \begin{array}{l} \textit{incoherent electric} \\ \text{image coefficients} \end{array} \right.$$

- Special case of two parallel perfect conducting plates
  - In the vertical plane we get the Laslett coefficient from the electric field



$$E_y = \frac{\lambda}{\pi\varepsilon_0 h^2}\left[(\bar{y}+y)\frac{\pi^2}{32} + (\bar{y}-y)\frac{\pi^2}{96}\right] \Longrightarrow \boxed{\varepsilon_1^y = \pi^2/48}$$

  - The horizontal force on the witness particle comes only from image charges and therefore it follows from the source free Gauss’ law

$$\vec{\nabla}\cdot\vec{E} = 0 \Longrightarrow \boxed{\varepsilon_1^x = -\varepsilon_1^y}$$

This holds for **all** geometries!

Electric image forces - coherent

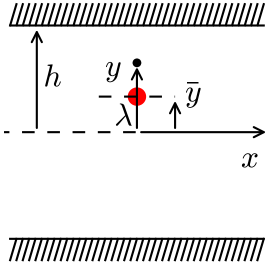
- In general the contribution from electric images to the *coherent* beam force for a variety of geometries can be expressed in terms of **"Laslett coefficients"**

$$\left.\frac{\partial \langle F_{\text{beam}}\rangle}{\partial x}\right|_{\bar{x}=0} + \left.\frac{\partial \langle F_{\text{beam}}\rangle}{\partial \bar{x}}\right|_{x=0} = \frac{e\lambda}{\pi\varepsilon_0}\frac{\xi_1^x}{h^2}$$

$$\left.\frac{\partial \langle F_{\text{beam}}\rangle}{\partial y}\right|_{\bar{y}=0} + \left.\frac{\partial \langle F_{\text{beam}}\rangle}{\partial \bar{y}}\right|_{y=0} = \frac{e\lambda}{\pi\varepsilon_0}\frac{\xi_1^y}{h^2}$$

$$\xi_1^x,\ \xi_1^y \ \ldots \left\{ \begin{array}{l} \textit{coherent electric} \\ \text{image coefficients} \end{array} \right.$$

- Special case of two parallel perfect conducting plates
  - In the vertical plane we get the Laslett coefficient from electrical field



$$E_y = \frac{\lambda}{\pi\varepsilon_0 h^2}\left[(\bar{y}+y)\frac{\pi^2}{32} + (\bar{y}-y)\frac{\pi^2}{96}\right] \Rightarrow \xi_1^y = \pi^2/16$$

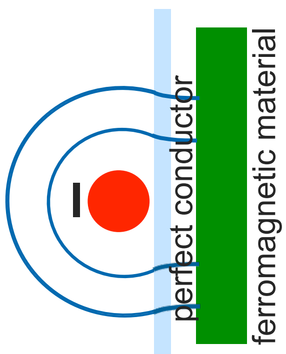
  - Due to the translational invariance in the horizontal plane in the case of the two parallel plates it follows that

$$\xi_1^x = 0$$

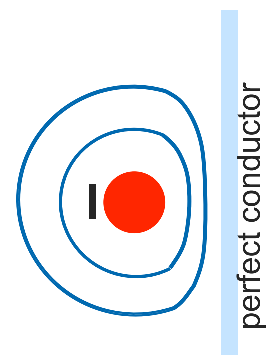
This is generally **not** the case!

Skin effect

- Static electric fields vanish inside a conductor for any finite conductivity, while static magnetic fields pass through (unless in case of very high permeability)
- This is no longer true for time varying fields, which can penetrate into the material in a region δw called skin depth
- The skin depth depends on the material properties and on frequency. Fields pass through the conductor wall if the skin depth is larger than the wall thickness Δ_w. This happens at low frequencies. At higher frequencies, for a good conductor $\delta w \ll \Delta w$ and both electric and magnetic fields vanish in the wall



If the fields penetrate and pass through the vacuum chamber, they can interact with bodies behind the chamber



If the skin depth is very small (rapidly varying fields), fields do not penetrate and the magnetic field lines are tangent to the surface.

Magnetic image forces

- Usually, the frequency beam spectrum is quite rich of harmonics, especially for bunched beams
- It is convenient to decompose the current into a dc component (proportional to the average line density) for which $\delta w >> \Delta w$ and an ac component (proportional to the peak line density) for which $\delta w << \Delta w$

$$I_{\mathrm{dc}} = \bar{\lambda}\beta c \qquad\qquad I_{\mathrm{ac}} = \hat{\lambda}\beta c$$

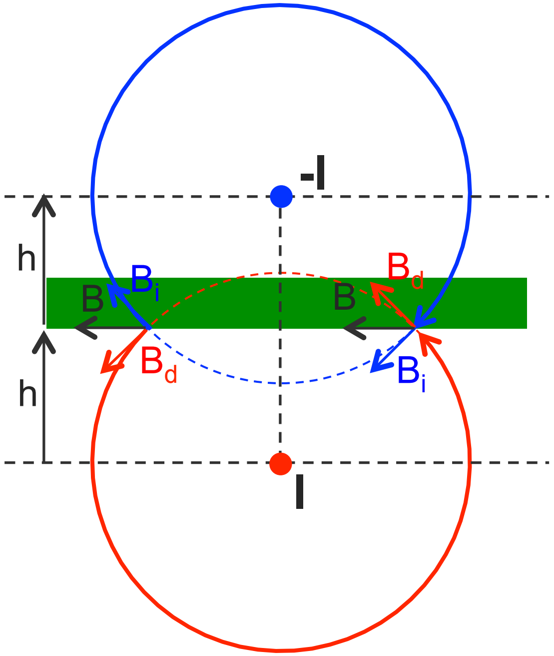
- The dc component of the magnetic field does not perceive the presence of the perfect conducting vacuum chamber and can reach the ferromagnetic poles of the magnets
- The ac component of the magnetic field does not penetrate the vacuum chamber and must thus be tangent to the pipe wall

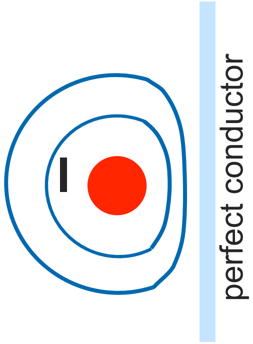
Magnetic image forces – ac components

- Let's look first at the **ac components** (i.e. non-penetratring fields)
  - The magnetic field is given by Stoke's law

$$B_\phi = \frac{\mu_0 \lambda \beta c}{2\pi r}$$

  - The boundary condition at a perfect conducting plate is $B_\perp = 0$ and can be satisfied by superposing an image current with opposite sign





Magnetic image forces – ac components

example: perfect conducting parallel plates

- Let's look first at the **ac components** (i.e. non-penetratring fields)
  - The magnetic field is given by Stoke's law

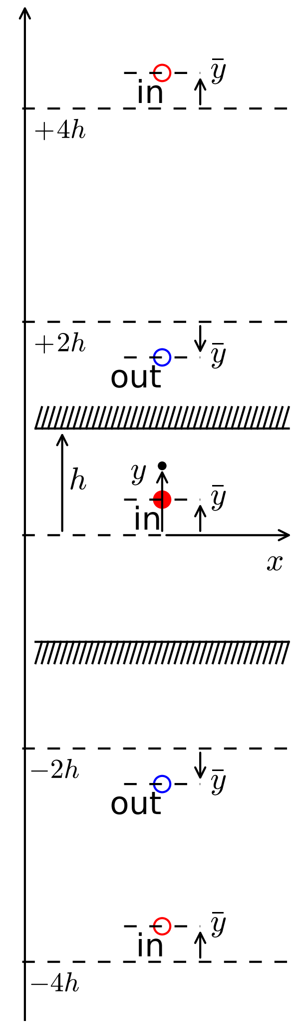
$$B_\phi = \frac{\mu_0 \lambda \beta c}{2\pi r}$$

- For the case of the 2 parallel perfect conducting plates we need an infinite series of image currents with alternating signs

$$\frac{F_y^{\rm mag}}{e} = -\frac{\mu_0 \lambda \beta^2 c^2}{2\pi} \left[+\frac{1}{2h-\bar{y}-y} - \frac{1}{2h+\bar{y}+y} + \frac{1}{6h-\bar{y}-y} - \frac{1}{6h+\bar{y}+y} + \ldots - \frac{1}{4h+\bar{y}-y} + \frac{1}{4h-\bar{y}+y} - \frac{1}{8h+\bar{y}-y} + \frac{1}{8h-\bar{y}+y} + \ldots \right]$$

$$\frac{F_y^{\rm mag}}{e} = -\frac{\lambda \beta^2}{\pi \varepsilon_0 h^2} \left[(\bar{y}+y)\frac{\pi^2}{32} + (\bar{y}-y)\frac{\pi^2}{96} \right]$$

.. apart from factor -β^2 this is the same result as for the electric field!



Magnetic image forces – ac components

- We have calculated that the contribution from ac magnetic components is the same as the one from the electric field apart from the factor $-\beta^2$
  - The magnetic field partially compensates the electric field like for direct space charge
  - This holds for all geometries (not only parallel plates)
- The contribution to the beam force is therefore written in terms of the Laslett coefficients of the *electric image charges*
  - Incoherent:

$$\left.\frac{\partial \langle F_{\rm beam}\rangle}{\partial x}\right|_{\bar{x}=0} = -\frac{e\lambda\beta^2}{\pi\varepsilon_0}\frac{\varepsilon_1^x}{h^2}$$

$$\left.\frac{\partial \langle F_{\rm beam}\rangle}{\partial y}\right|_{\bar{y}=0} = -\frac{e\lambda\beta^2}{\pi\varepsilon_0}\frac{\varepsilon_1^y}{h^2}$$

$\varepsilon_1^x,\ \varepsilon_1^y\ \ldots \left\{ \begin{array}{l} \textit{incoherent electric} \\ \text{image coefficients} \end{array} \right.$

  - Coherent:

$$\left.\frac{\partial \langle F_{\rm beam}\rangle}{\partial x}\right|_{\bar{x}=0} + \left.\frac{\partial \langle F_{\rm beam}\rangle}{\partial \bar{x}}\right|_{x=0} = -\frac{e\lambda\beta^2}{\pi\varepsilon_0}\frac{\xi_1^x}{h^2}$$

$$\left.\frac{\partial \langle F_{\rm beam}\rangle}{\partial y}\right|_{\bar{y}=0} + \left.\frac{\partial \langle F_{\rm beam}\rangle}{\partial \bar{y}}\right|_{y=0} = -\frac{e\lambda\beta^2}{\pi\varepsilon_0}\frac{\xi_1^y}{h^2}$$

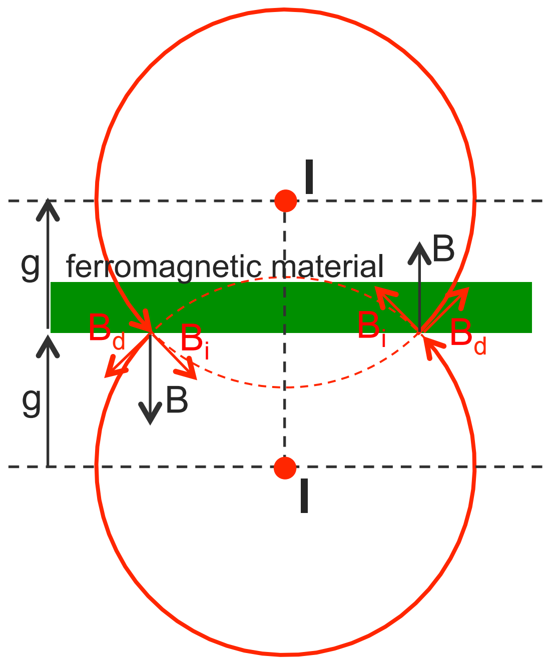
$\xi_1^x,\ \xi_1^y\ \ldots \left\{ \begin{array}{l} \textit{coherent electric} \\ \text{image coefficients} \end{array} \right.$

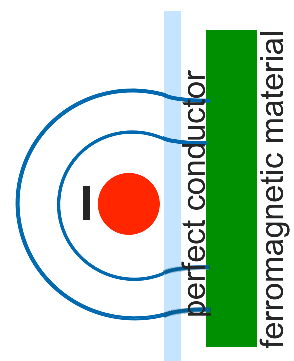
Magnetic image forces – dc components

- The dc components of the magnetic field are not affected by the perfect conducting wall of the vacuum chamber (i.e. the fields are penetrating)
- If the vacuum chamber is surrounded by a magnet of the accelerator the boundary condition is given by the ferromagnetic material of the magnet yoke
- The boundary condition at a ferromagnetic surface is satisfied by adding an image current with the same direction as the beam current

$$B_{\perp} = \text{continous}$$

$$B_{\|} = 0$$





Magnetic image forces – dc components

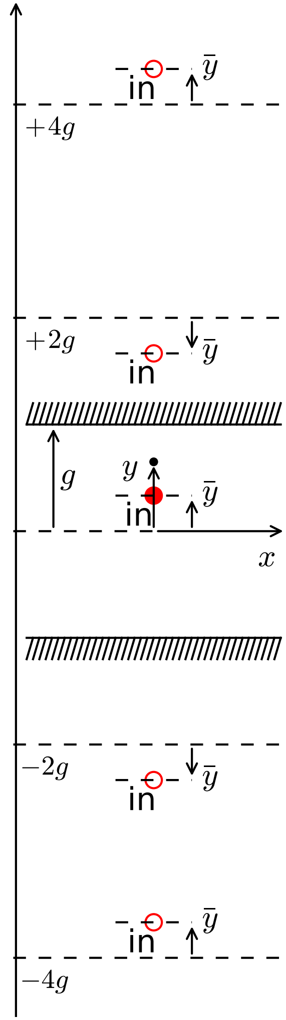
example: ferromagnetic parallel plates

- We look now at the case of two ferromagnetic parallel plates spaced by a distance of 2g
  - The boundary condition is satisfied by superposing an infinite number of image currents all flowing in the same direction as the beam current and so the force on the witness particle becomes

$$\frac{F_y^{\rm mag}}{e} = \frac{\mu_0 \lambda \beta^2 c^2}{2\pi}\left[+\frac{1}{2g-\bar{y}-y} - \frac{1}{2g+\bar{y}+y} + \frac{1}{6g-\bar{y}-y} - \frac{1}{6g+\bar{y}+y} + \dots \right.$$
$$\left. + \frac{1}{4g+\bar{y}-y} - \frac{1}{4g-\bar{y}+y} + \frac{1}{8g+\bar{y}-y} - \frac{1}{8g-\bar{y}+y} + \dots \right]$$

  - Following the same steps as we did for the electric field …

$$\frac{F_y^{\rm mag}}{e} = +\frac{\lambda \beta^2}{\pi \varepsilon_0 g^2}\left[(\bar{y}+y)\frac{\pi^2}{32} - (\bar{y}-y)\frac{\pi^2}{96}\right]$$



Magnetic image forces – dc components

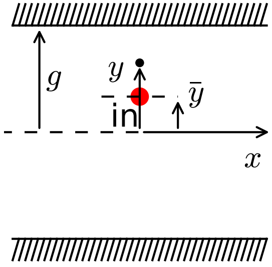
- As before, we express the **incoherent** contributions to the beam force in terms of the Laslett coefficients for dc magnetic images

$$\left.\frac{\partial \langle F_{\text{beam}} \rangle}{\partial x}\right|_{\bar{x}=0} = +\frac{e\lambda\beta^2}{\pi\varepsilon_0}\frac{\varepsilon_2^x}{g^2}$$

$$\left.\frac{\partial \langle F_{\text{beam}} \rangle}{\partial y}\right|_{\bar{y}=0} = +\frac{e\lambda\beta^2}{\pi\varepsilon_0}\frac{\varepsilon_2^y}{g^2}$$

$\varepsilon_2^x,\ \varepsilon_2^y \ \ldots$ { *incoherent dc magnetic* image coefficients

- Special case of two parallel ferromagnetic plates
  - We know the force in the vertical direction



$$\frac{F_y^{\text{mag}}}{e} = +\frac{\lambda\beta^2}{\pi\varepsilon_0 g^2}\left[(\bar{y}+y)\frac{\pi^2}{32} - (\bar{y}-y)\frac{\pi^2}{96}\right] \Longrightarrow \varepsilon_2^y = \pi^2/24$$

  - The horizontal force on the witness particle comes only from image currents and therefore it follows from the source free Stoke's law

$$\vec{\nabla} \times \vec{B} = 0 \Longrightarrow \varepsilon_2^x = -\varepsilon_2^y$$

This holds for **all** geometries!

Magnetic image forces – dc components

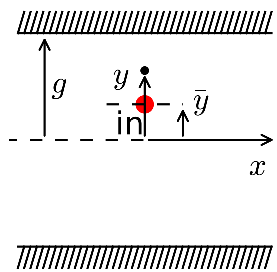
- Finallly we come to the **coherent** contributions from dc magnetic images

$$\left.\frac{\partial \langle F_{\text{beam}} \rangle}{\partial x}\right|_{\bar{x}=0} + \left.\frac{\partial \langle F_{\text{beam}} \rangle}{\partial \bar{x}}\right|_{x=0} = +\frac{e\lambda\beta^2}{\pi\varepsilon_0}\frac{\xi_2^x}{g^2}$$

$$\left.\frac{\partial \langle F_{\text{beam}} \rangle}{\partial y}\right|_{\bar{y}=0} + \left.\frac{\partial \langle F_{\text{beam}} \rangle}{\partial \bar{y}}\right|_{y=0} = +\frac{e\lambda\beta^2}{\pi\varepsilon_0}\frac{\xi_2^y}{g^2}$$

$\xi_2^x,\ \xi_2^y\ \ldots$ { *coherent dc magnetic* image coefficients

- Special case of two parallel ferromagnetic plates
  - We know the force in the vertical direction



$$\frac{F_y^{\text{mag}}}{e} = +\frac{\lambda\beta^2}{\pi\varepsilon_0 g^2}\left[(\bar{y}+y)\frac{\pi^2}{32} - (\bar{y}-y)\frac{\pi^2}{96}\right] \Longrightarrow \boxed{\xi_2^y = \pi^2/16}$$

  - Due to the translational invariance in the horizontal plane in the case of the two parallel plates it follows that

$$\boxed{\xi_2^x = 0}$$

This is generally **not** the case!

Laslett coefficients

- Now we have gathered all coherent and incoherent contributions to the beam force and expressed them in terms of Laslett coefficients
- The Laslett coefficients have been computed* also for other geometries such as circular and elliptical beam pipes
  - Note: they are always defined with respect to the *vertical half gap h or g*

Laslett coefficients	Circular ($w = h$)	Elliptical (e.g. $w = 2h$)	Parallel plates ($h/w = 0$)
ε_1^x	0	-0.172	$-\pi^2/48$
ε_1^y	0	$+0.172$	$+\pi^2/48$
ξ_1^x	$+1/2$	0.083	0
ξ_1^y	$+1/2$	0.55	$+\pi^2/16$
ε_2^x	$-\pi^2/24$	$-\pi^2/24$	$-\pi^2/24$
ε_2^y	$+\pi^2/24$	$+\pi^2/24$	$+\pi^2/24$
ξ_2^x	0	0	0
ξ_2^y	$+\pi^2/16$	$+\pi^2/16$	$+\pi^2/16$

Assuming parallel plates for the ferro-magnetic boundary for all geometries … (rows ε_2^x to ξ_2^y)

* L. J. Laslett, LBL Document PUB-616, 1987, vol III

Overview of force contributions

- The indirect space charge contributions from the different image charges and currents to the coherent and incoherent beam force are summarized below
  - As mentioned before, the magnetic force due to image currents in the vacuum chamber (ac components) are described by the electric image coefficients
  - The ac coherent component corresponds to betatron beam oscillations

Beam force components	Images in vacuum chamber		Images in pole faces	Comments
	electric	magnetic	magnetic	
$\left.\frac{\partial\langle F_{\text{beam}}\rangle}{\partial y}\right\|_{\bar{y}=0}$	$\frac{\varepsilon_1^y}{h^2}$	$-\beta^2\frac{\varepsilon_1^y}{h^2}$	$\beta^2\frac{\varepsilon_2^y}{g^2}$	incoherent, dc coherent
$\left.\frac{\partial\langle F_{\text{beam}}\rangle}{\partial y}\right\|_{\bar{y}=0}+\left.\frac{\partial\langle F_{\text{beam}}\rangle}{\partial \bar{y}}\right\|_{y=0}$	$\frac{\xi_1^y}{h^2}$	$-\beta^2\frac{\xi_1^y}{h^2}$	$\beta^2\frac{\xi_2^y}{g^2}$	coherent
$\left.\frac{\partial\langle F_{\text{beam}}\rangle}{\partial \bar{y}}\right\|_{y=0}$		$-\beta^2\frac{\xi_1^y-\varepsilon_1^y}{h^2}$	$\beta^2\frac{\xi_2^y-\varepsilon_2^y}{g^2}$	ac coherent

Indirect space charge: incoherent tune shift

- We found the expression for the coherent tune shift as function of the beam force as

$$\Delta Q_y^{\text{inc}} = \frac{R^2}{2Q_{y0}\beta^2 c^2 m\gamma}\left(\frac{\partial\langle F_{\text{beam}}\rangle}{\partial y}\Big|_{\bar{y}=0}\right) = \frac{R\langle\beta_y\rangle}{2\beta^2 c^2 m\gamma}\left(\frac{\partial\langle F_{\text{beam}}\rangle}{\partial y}\Big|_{\bar{y}=0}\right)$$

  - In the second step the tune shift is written in terms of the average beta function instead of the betatron tune (as given by the smooth approximation) – like this we can refine our model and account for different geometries around the machine

- Collecting all contributions to the incoherent incoherent tune shift we find for the general case of bunched beams
  - For coasting beams the peak line density is equal to the average line density
  - The F factor corresponds to the ratio of the circumference surrounded by ferromagnetic material

$$\Delta Q_{x,y}^{\text{inc}} = -\frac{2\langle\beta_{x,y}\rangle r_0 R}{e\beta^2\gamma}\left[\underbrace{\frac{\varepsilon_1^{x,y}}{h^2}\hat{\lambda}}_{\text{electric image}} - \underbrace{\beta^2\frac{\varepsilon_1^{x,y}}{h^2}\left(\hat{\lambda}-\bar{\lambda}\right)}_{\text{ac magnetic image from bunching}} + \underbrace{\mathcal{F}\beta^2\frac{\varepsilon_2^{x,y}}{g^2}\bar{\lambda}}_{\text{magnetic image in magnet poles}}\right]$$

Indirect space charge: coherent tune shift

- The general expression of the coherent tune shift is obtained as

$$\Delta Q_y^{\text{coh}} = \frac{R\langle\beta_y\rangle}{2\beta^2 c^2 m\gamma}\left(\frac{\partial\langle F_{\text{beam}}\rangle}{\partial y}\bigg|_{\bar{y}=0} + \frac{\partial\langle F_{\text{beam}}\rangle}{\partial \bar{y}}\bigg|_{y=0}\right)$$

- Here we need to consider two cases
  - Betatron oscillations are of such low frequency that the induced magnetic field can penetrate through the vacuum chamber

$$\Delta Q_{x,y}^{\text{coh}} = -\frac{2\langle\beta_{x,y}\rangle r_0 R}{e\beta^2\gamma}\left[\underbrace{\frac{\xi_1^{x,y}}{h^2}\hat{\lambda}}_{\substack{\text{electric}\\ \text{image}}} - \underbrace{\beta^2\frac{\xi_1^{x,y}}{h^2}\left(\hat{\lambda}-\bar{\lambda}\right)}_{\substack{\text{ac magnetic image}\\ \text{from bunching}}} + \underbrace{\mathcal{F}\beta^2\frac{\xi_2^{x,y}}{g^2}\bar{\lambda}}_{\substack{\text{magnetic image}\\ \text{in magnet poles}}}\right]$$

  - Magnetic fields from both betatron oscillations and longitudinal bunching cannot penetrate the vacuum chamber

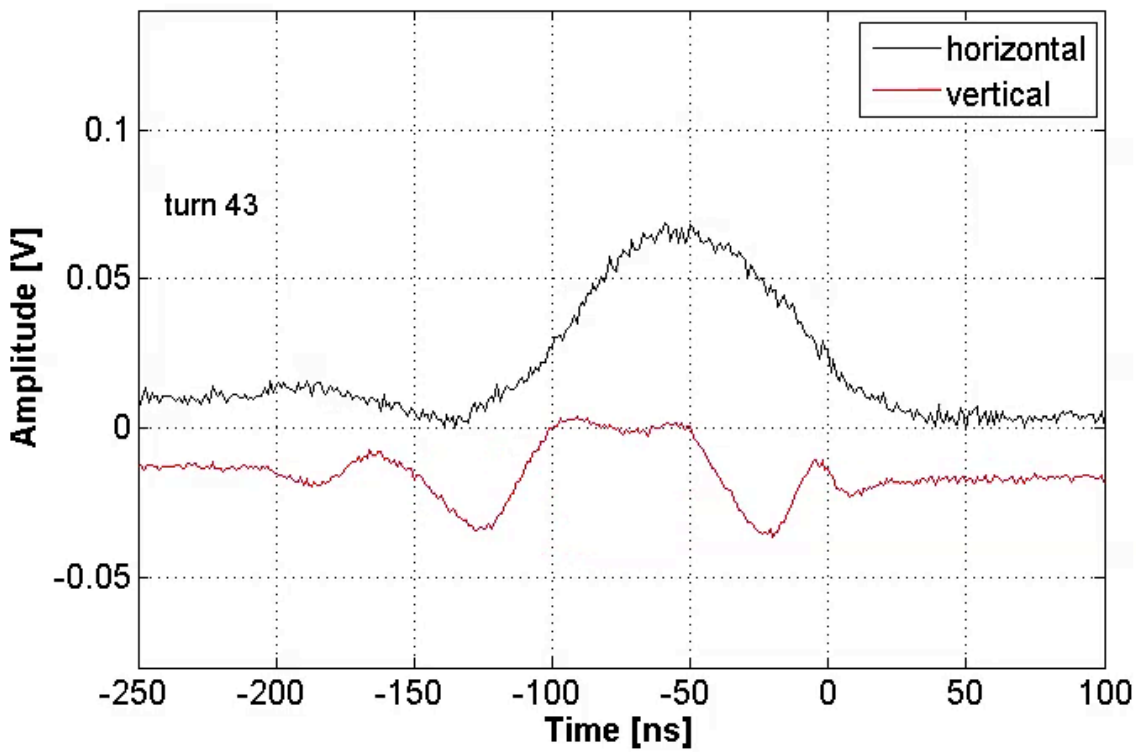
$$\Delta Q_{x,y}^{\text{coh}} = -\frac{2\langle\beta_{x,y}\rangle r_0 R}{e\beta^2\gamma}\left[\underbrace{\frac{\xi_1^{x,y}}{h^2}\hat{\lambda}}_{\substack{\text{electric}\\ \text{image}}} - \underbrace{\beta^2\frac{\xi_1^{x,y}}{h^2}\left(\hat{\lambda}-\bar{\lambda}\right)}_{\substack{\text{ac magnetic image}\\ \text{from bunching}}} - \underbrace{\beta^2\frac{\xi_1^{x,y}-\varepsilon_1^{x,y}}{h^2}\bar{\lambda}}_{\substack{\text{ac magnetic image}\\ \text{from transverse motion}}} + \underbrace{\mathcal{F}\beta^2\frac{\varepsilon_2^{x,y}}{g^2}\bar{\lambda}}_{\substack{\text{magnetic image}\\ \text{in magnet poles}}}\right]$$

Remarks about indirect space charge

- The incoherent tune shifts have always opposite signs in the two planes
- The coherent tune shifts are never positive
- Since we have used the smooth approximation the indirect space charge tune shift is proportional to average β-function
- The formulas can be easily generalized to account for the β-function variation and different vacuum chamber geometries around the machine
- The $1/\gamma$ dependence of the indirect space charge tune shift stems from the fact that the charged particles induce the electrostatic field and thus generate a force proportional to their number, but independent of their mass, whereas the deflection of the beam by this force is inversely proportional to their mass $m_0\gamma$

Example: PS injection oscillations

- Intra-bunch motion is observed when injecting a single bunch into the PS
  - This motion is more pronounced in the vertical plane
  - The intra-bunch frequency increases turn by turn

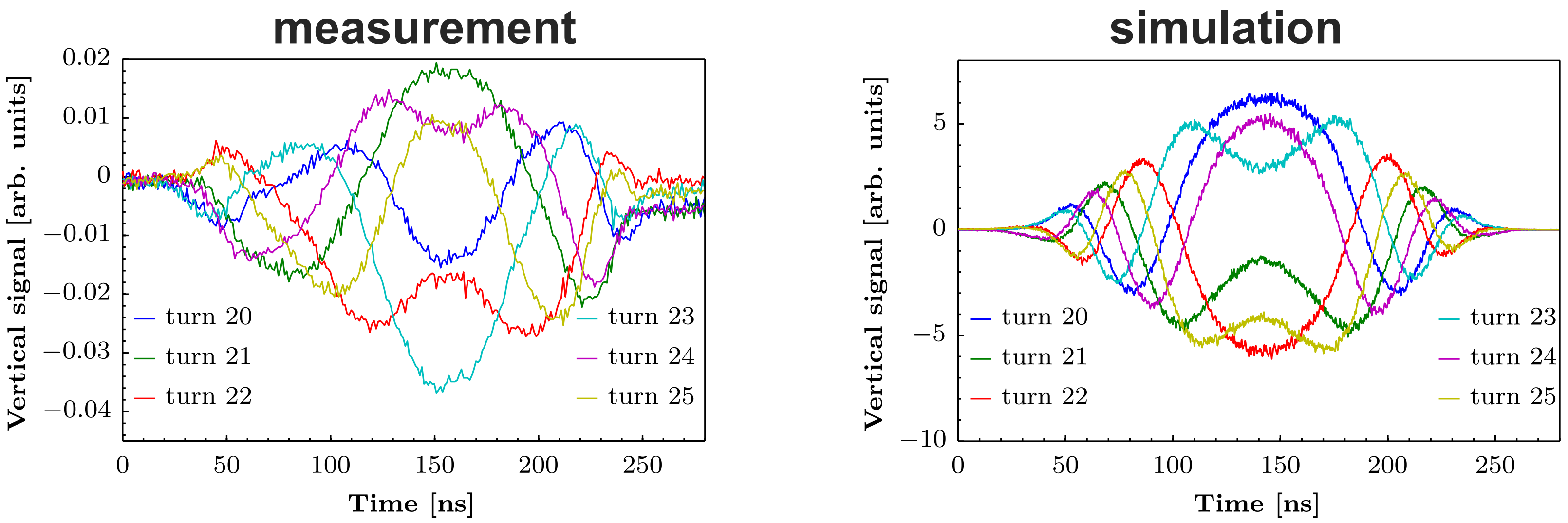


transverse position along the bunch measured with a wide-band pick-up (bunch length about 200 ns)

A. Huschauer et. al

Example: PS injection oscillations

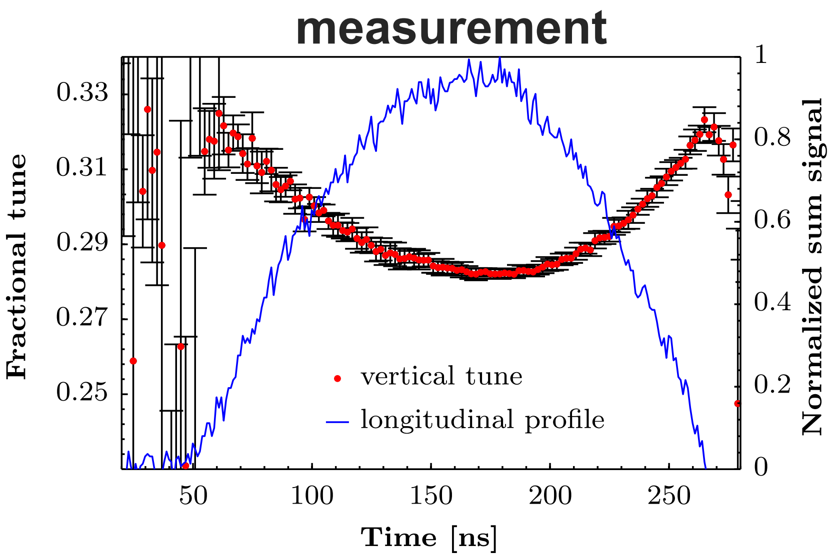
- The observed intra-bunch motion was reproduced with an amazing precision with multi-particle simulations (HEADTAIL code) including the indirect space charge effect taking

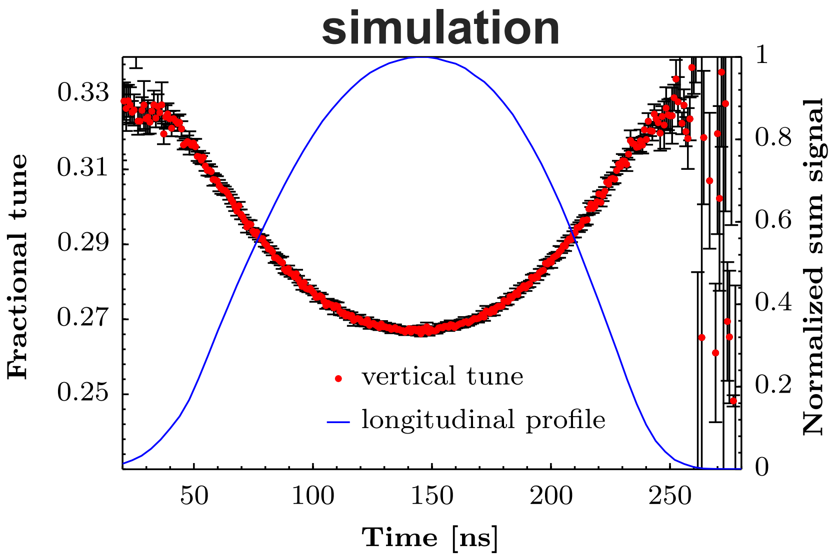


A. Huschauer et. al

Example: PS injection oscillations

- The observed intra-bunch motion was reproduced with an amazing precision with multi-particle simulations (HEADTAIL code) including the indirect space charge effect taking
- It was understood from simulations that the observed intra-bunch motion was induced by the beam injected off-center in combination with the indirect space charge effect, which causes a tune shift along the bunch proportional to the local charge density





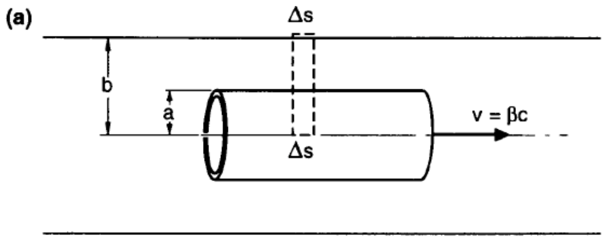
A. Huschauer et. al

Outline

1. Introduction
   - Accelerator concepts
   - Coordinate systems and phase space
   - Transverse single particle dynamics
   - Longitudinal single particle dynamics
   - Collective effects

2. Space charge
   - Direct space charge (transverse)
   - Indirect space charge (transverse)
   - Longitudinal space charge

Longitudinal space charge



- We look at a longitudinally non-uniform distribution $\lambda(s\text{-}\beta ct)$ in a perfect conducting round beam pipe with radius *b*
- Let the beam have a ring shaped distribution with radius *a*
  - If the bunch length is much longer than b/γ, the transverse electric and magnetic fields are determined by the local line density and approximately given by

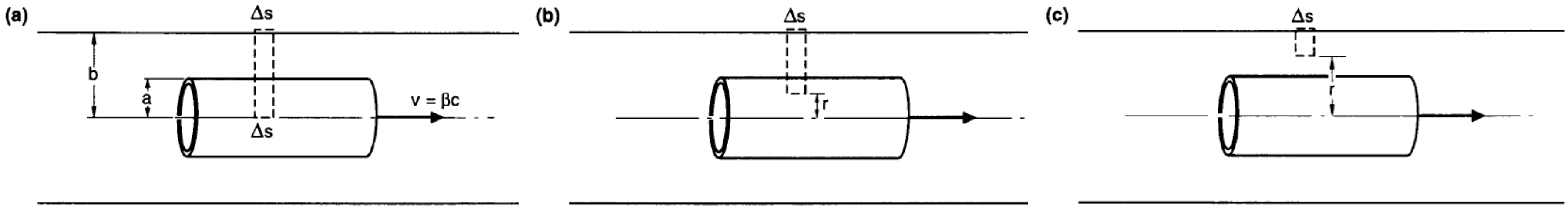
$$E_r(r, s-\beta ct) = \frac{cB_\phi(r, s-\beta ct)}{\beta} = \begin{cases} 0 & \text{if} \quad 0 < r < a \\ \frac{\lambda(s-\beta ct)}{2\pi\varepsilon_0 r} & \text{if} \quad a < r < b \end{cases}$$

- We calculate the longitudinal electric field using

3rd Maxwell equation
(Faraday-Neumann law)

$$\oint_{\partial\Omega} \vec{E} \cdot d\vec{l} = -\frac{d}{dt} \int_{\Omega} \vec{B} \cdot d\vec{S}$$

Longitudinal space charge



$$E_s(r,s-\beta ct)\Delta s+\int_r^b E_r(r',s-\beta ct)dr'+\int_b^r E_r(r',s+\Delta s-\beta ct)dr'=-\frac{d}{dt}\int_r^b B_\phi(r',s-\beta ct)dr'\Delta s$$

$$E_s(r,s-\beta ct)+\frac{1}{2\pi\varepsilon_0}\frac{\lambda(s-\beta ct)-\lambda(s+\Delta s-\beta ct)}{\Delta s}\int_r^b\frac{dr'}{r'}=-\frac{\beta}{2\pi\varepsilon_0 c}\frac{\partial\lambda(s-\beta ct)}{\partial t}\int_r^b\frac{dr'}{r'}$$

- We integrate, take the limit Δs → 0, and look at three integration paths
  - Path (a) and (b) give the same result since the transverse fields vanish in the cylinder
  - Path (c) gives the result for r > a

$$E_s(r,s-\beta ct)=\begin{cases}-\frac{1}{2\pi\varepsilon_0\gamma^2}\lambda'(s-\beta ct)\log\frac{b}{a} & \text{if } 0<r<a\\ -\frac{1}{2\pi\varepsilon_0\gamma^2}\lambda'(s-\beta ct)\log\frac{b}{r} & \text{if } a<r<b\end{cases}$$

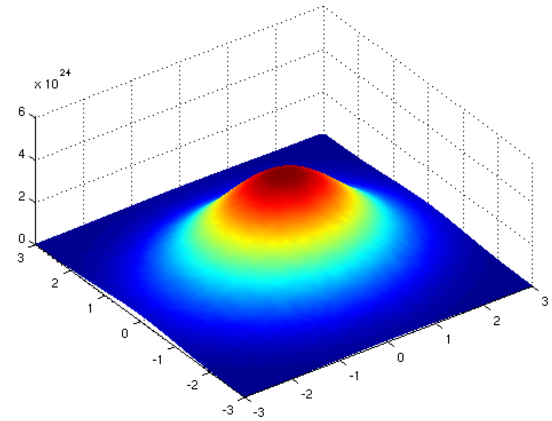
3rd Maxwell equation
(Faraday-Neumann law)

$$\oint_{\partial\Omega}\vec{E}\cdot d\vec{l}=-\frac{d}{dt}\int_\Omega\vec{B}\cdot d\vec{S}$$

Longitudinal space charge

- We have found the space charge longitudinal electric field for a cylindrically symmetric ring shaped distribution
- Let's assume now that we have a beam with a normalized radial distribution n(r)

$$\int_0^\infty 2\pi r n(r) dr = 1$$



- Then for the linearity of Maxwell's equations, we can calculate its longitudinal space charge electric field as the superposition of the contributions given by the various rings composing the actual distribution

$$E_s(r, s-\beta ct) = -\frac{1}{2\pi\varepsilon_0\gamma^2}\lambda'(s-\beta ct)\left[\log\frac{b}{r} + \int_r^b 2\pi r' n(r')\log\left(\frac{r}{r'}\right) dr'\right]$$

Longitudinal space charge

$$E_s(r, s-\beta ct) = -\frac{1}{2\pi\varepsilon_0\gamma^2}\lambda'(s-\beta ct)\left[\log\frac{b}{r} + \int_r^b 2\pi r' n(r')\log\left(\frac{r}{r'}\right)dr'\right]$$

- Longitudinal space charge has the following interesting dependencies:
  - It decreases with energy like $1/\gamma^2$. Therefore it vanishes in the ultrarelativistic limit
  - It is proportional to the opposite of the derivative of the line density $-\lambda'$. This can be understood intuitively because it must be directed from a region with higher charge density to a region with lower charge density (i.e. it pushes with the opposite of the gradient of the line charge)
  - Space charge would then spread out charge bumps. However, remember that only below transition energy, accelerated particles go faster and space charge has this smoothing action. Above transition, accelerated particles take a longer time to go around the accelerator and density peaks can be enhanced. This is the origin of the so-called **negative mass instability**. Momentum spread (unbunched beams) or synchrotron motion (bunched beams) can usually stabilize this effect.
- Note about this formula
  - For r=b the longitudinal electrical field vanishes as it should (perfect conducting pipe)
  - It diverges for $b \rightarrow \infty$ because the analysis breaks down if the bunch length $\leq b/\gamma$

Longitudinal space charge

example: transverse uniform distribution

$$E_s(r, s-\beta ct) = -\frac{1}{2\pi\varepsilon_0\gamma^2}\lambda'(s-\beta ct)\left[\log\frac{b}{r} + \int_r^b 2\pi r' n(r')\log\left(\frac{r}{r'}\right)dr'\right]$$

- We look at the case of a normalized transverse uniform distribution n

$$n = \begin{cases} \frac{1}{\pi a^2} & \text{if} \quad 0 < r < a \\ 0 & \text{if} \quad a < r < b \end{cases}$$

  - The resulting longitudinal electric field is obtained as

$$E_s(r, s-\beta ct) = -\frac{1}{2\pi\varepsilon_0\gamma^2}\lambda'(s-\beta ct)\begin{cases} \log\frac{b}{a} + \frac{1}{2} - \frac{r^2}{2a^2} & \text{if} \quad 0 < r < a \\ \log\frac{b}{r} & \text{if} \quad a < r < b \end{cases}$$

  - The longitudinal field on axis (r=0) is given by

$$E_s(0, s-\beta ct) = -\frac{1}{2\pi\varepsilon_0\gamma^2}\lambda'(s-\beta ct)\left(\log\frac{b}{a} + \frac{1}{2}\right)$$

Synchrotron tune shift due to space charge

- We consider the case of a parabolic bunch inside a stationary single RF bucket
  - Longitudinal equations of motion (see Kevin's presentation)

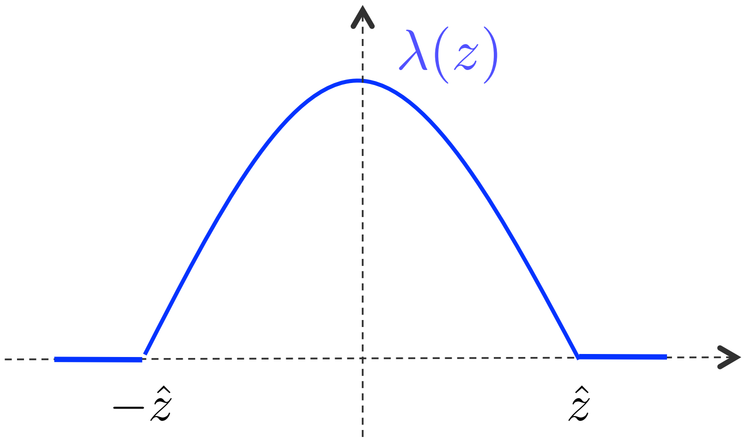
$$\begin{cases} \dot{z} = -\eta\beta c\delta \\ \dot{\delta} = \underbrace{\frac{eV}{p_0 C}\sin\left(\frac{hz}{R}\right)}_{\approx \frac{eVh}{p_0 CR} z} - \frac{eg\lambda'(z)}{2\pi\varepsilon_0\gamma^2 p_0} \end{cases} \quad \text{with} \quad g = \log\frac{b}{a} + \frac{1}{2}$$

longitudinal space charge term

$\approx \frac{eVh}{p_0 CR} z$ (small amplitude approximation)

  - Line density function for a parabolic distribution

$$\lambda(z) = \begin{cases} \frac{3N_b e}{4\hat{z}^3}\left(\hat{z}^2 - z^2\right) & \text{if} \quad |z| \leq \hat{z} \\ 0 & \text{if} \quad |z| > \hat{z} \end{cases}$$



Synchrotron tune shift due to space charge

- While the space charge force is already linear with the chosen bunch line density, we linearize the RF force around z=0

$$\ddot{z} + \underbrace{\left(\frac{\eta e V h \beta c}{p_0 C R} + \frac{3e^2 g N_b \eta \beta c}{4\pi\varepsilon_0 \gamma^2 \hat{z}^3 p_0} \right)}_{\omega_s^2 = \omega_0^2 Q_s^2 \approx \omega_0^2 \cdot (Q_{s0} + \Delta Q_s)^2} z = 0$$

  - From this we can calculate the synchrotron tune shift due to space charge in the radial center of a transverse Parabolic bunch

$$\Delta Q_s \approx \frac{3e^2 g N_b \eta R^2}{8\pi\varepsilon_0 \beta^2 \gamma^2 \hat{z}^3 E_0 Q_{s0}}$$

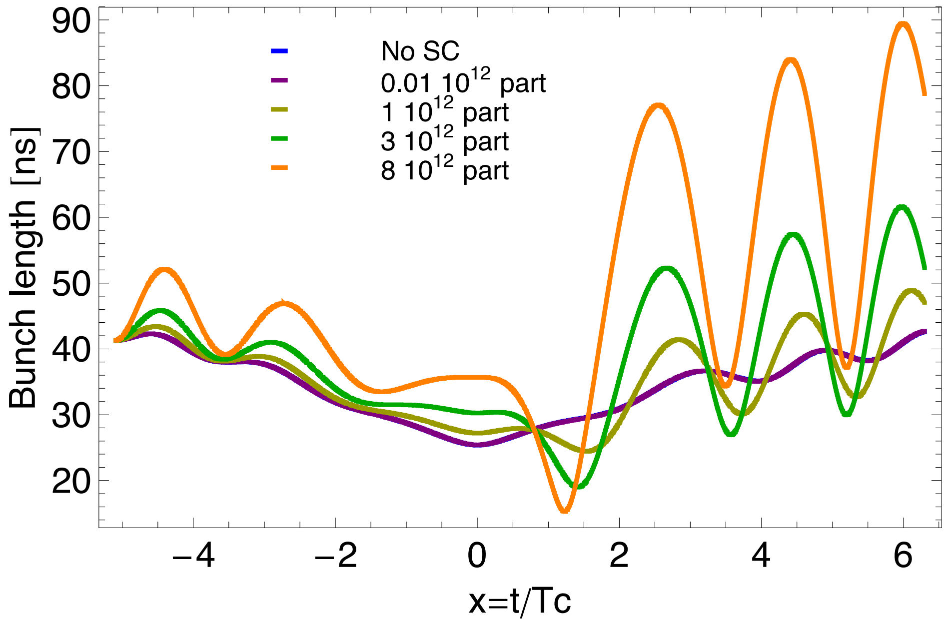
Synchrotron tune shift due to space charge

$$\Delta Q_s \approx \frac{3e^2 g N_b \eta R^2}{8\pi\varepsilon_0 \beta^2 \gamma^2 \hat{z}^3 E_0 Q_{s0}}$$

… synchrotron tune shift due to space charge in the radial center of a transverse Parabolic bunch

- The tune shift is proportional to number of particles in the bunch N_b (linear)
- Beam energy dependence
  - Explicit inverse quadratic dependence on β and γ
  - There is another energy dependence in η. In particular the shift changes sign below and above transition. The tune shift is negative below transition (η<0), where space charge is defocusing. It is positive above transition (η>0), where space charge adds up to the external voltage and contributes with an extra focusing.
  - There is another inverse dependence on γ in E_0.
- Tune shift proportional to Machine radius (quadratic)
- Bunch length dependence (inverse cubic)
  - Attention must be paid here to the fact that the bunch length itself depends upon the strength of space charge….

Example: PS transition crossing



- The longitudinal space charge force changes sign when crossing transition
  - Quadrupolar oscillations are excited due to the sudden mismatch of the beam distribution at transtion
  - This effect gets stronger with increasing space charge force
  - To avoid the mismatch, a jump of the RF voltage needs to be programmed similar to the phase jump

Summary

- Direct space charge
  - interaction of the bunch particles with the self induced electro-magnetic fields in free space
  - results in an incoherent tune shift (or spread)
  - the space charge force along a bunch is modulated with the local line density along the bunch and this results in an additional tune spread
  - decreases with energy like $\beta^{-1}\gamma^{-2}$
  - is a typical performance limitation for low energy machines
- Indirect space charge
  - interaction with image charges and currents induced in perfect conducting walls and ferromagnetic materials close to the beam pipe
  - results in incoherent and coherent tune shifts (or spreads), some of which are proportional to the average line density and others to the local line density
  - The contributions to the coherent and incoherent tune shifts for different standard geometries are expressed in terms of Laslett coefficients
  - decreases with energy like $\beta^{-2}\gamma^{-1}$

Summary

- Longitudinal space charge
  - Is the interaction of particles with the self field and the image charges induced in the vacuum chamber
  - Is proportional to the derivative of the line density
  - Is defocusing below and focusing above transition
  - Can lead to a mismatch of the particle distribution in the longitudinal phase space after transition crossing and to the negative mass instability above transition