CERN

U.S. Particle Accelerator School
Education in Beam Physics and Accelerator Technology

Collective effects in Beam Dynamics

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USPAS, one-week course, 19-24 January, 2015
http://uspas.fnal.gov/index.shtml

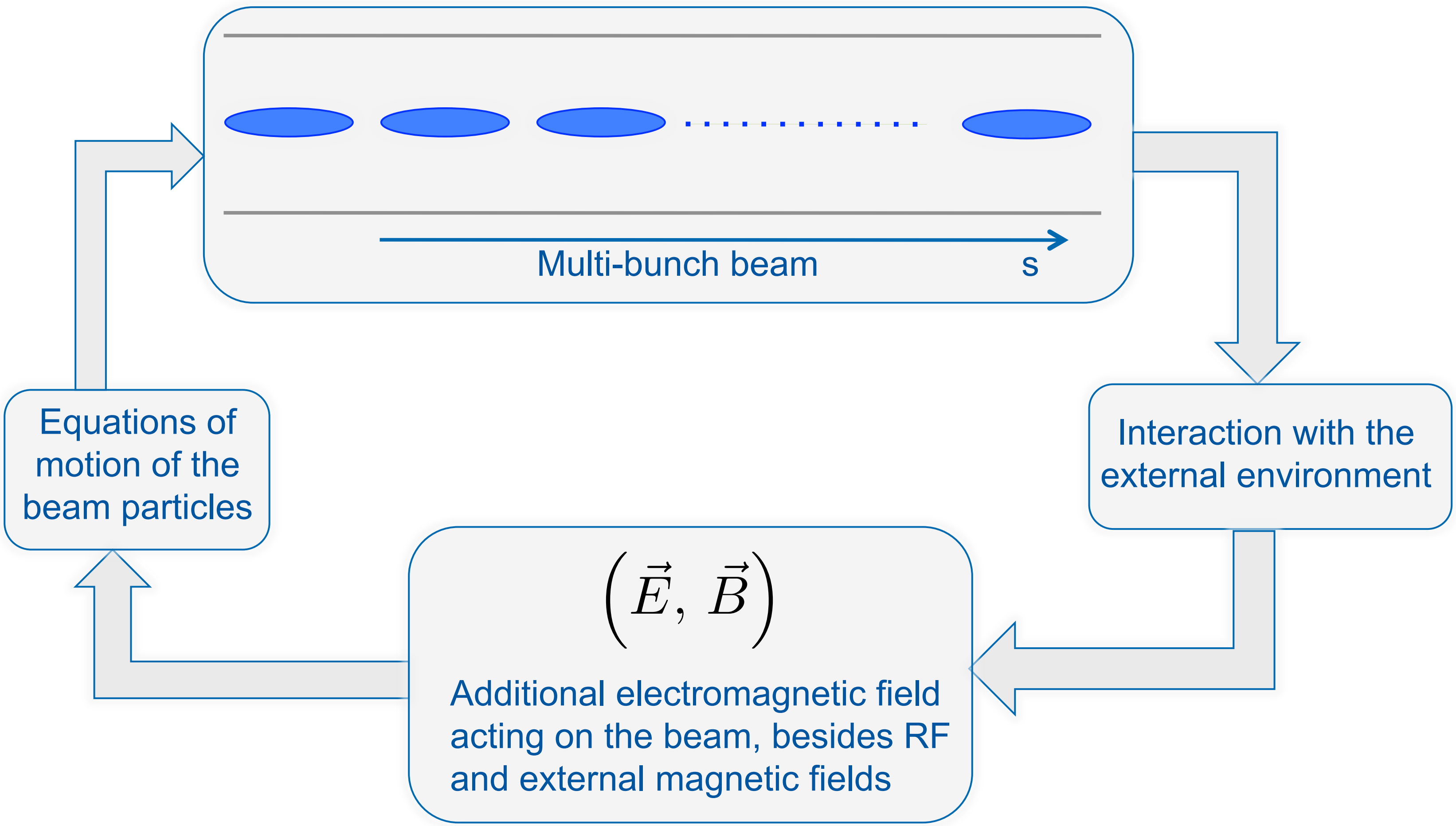
Outline

1. Introductory concepts
   - Collective effects
   - Transverse single particle dynamics including systems of many non-interacting particles
   - Longitudinal single particle dynamics including systems of many non-interacting particles
2. Space charge
   - Direct space charge (transverse)
   - Indirect space charge (transverse)
   - Longitudinal space charge

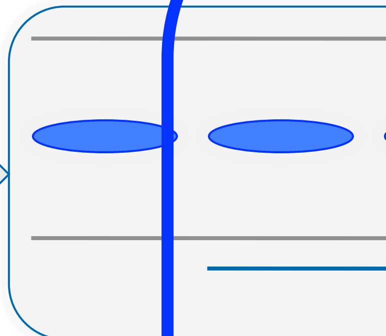
Outline

3. Wake fields and impedance
   - Wake function and wake potential
   - Definition of beam coupling impedance
   - Examples – resonators and resistive wall
   - Energy loss
   - Impedance model of a machine
4. **Instabilities – few-particle model**
   - Equations of motion
   - Longitudinal plane: Robinson instability
   - Transverse plane: rigid bunch instability, strong head-tail instability, head-tail instability

Reminder


Multi-bunch beam
s
Interaction with the external environment
$\left(\vec{E},\, \vec{B}\right)$
Additional electromagnetic field acting on the beam, besides RF and external magnetic fields
Equations of motion of the beam particles

Reminder



Equations of motion of the beam particles

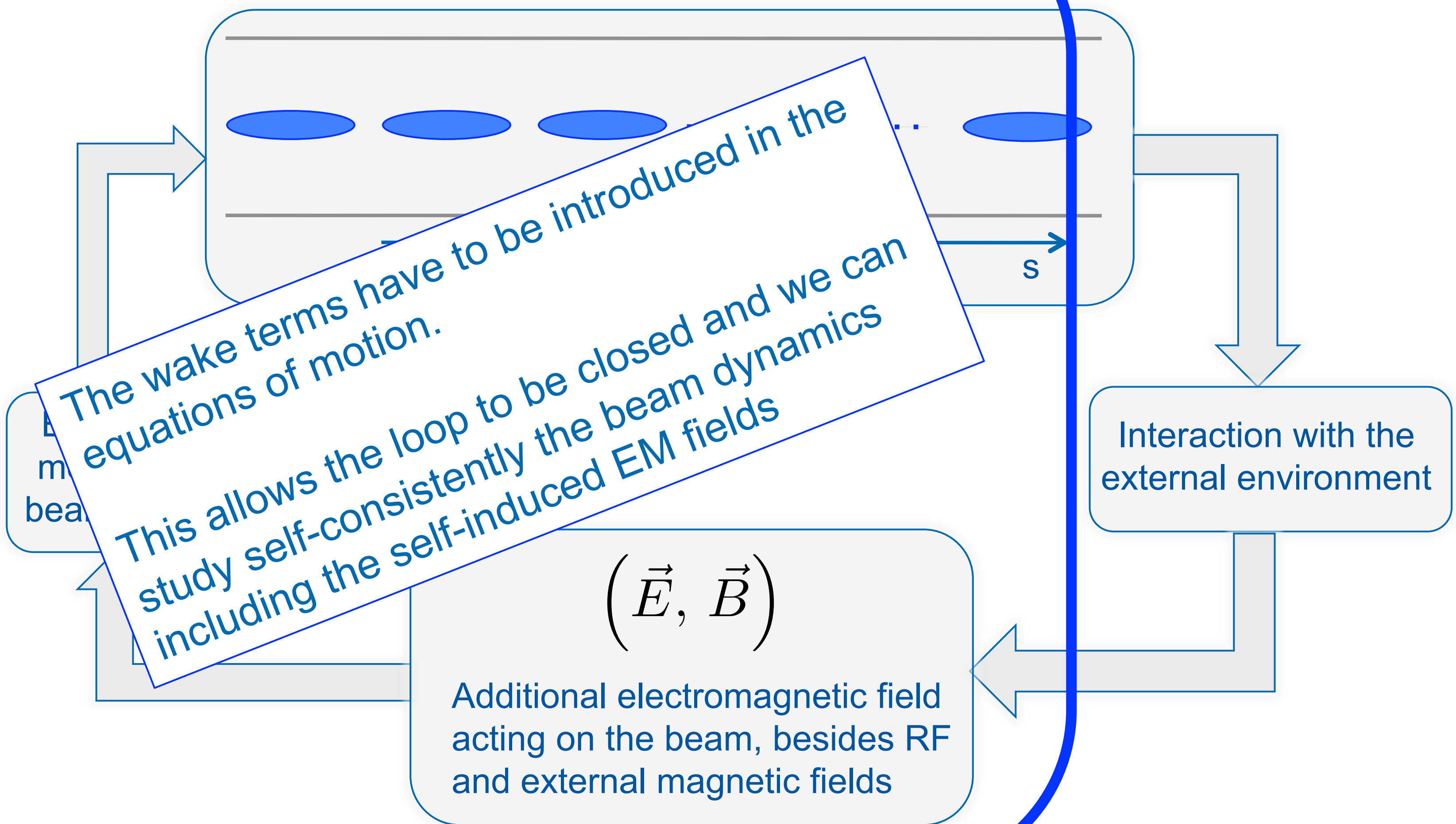
...with the ...ronment

Additi... acting ... and external magnetic fields

Wakes and impedances

- Particles traveling through accelerator elements create trailing electromagnetic fields whose effect is modeled through the **wake fields**
- Wakes/impedances are computed
  - (Semi-)Analytically – e.g. resistive wall for simple geometries
  - Numerically by means of EM programs (e.g. CST Studio Suite)
  - By means of bench measurements
- In a beam, all particles both create wakes and suffer from them
  - The beam suffers **energy loss** in the creation of the wake and its interaction with it
- Wakes/impedances from different elements are combined to create the **'impedance model'** of a given accelerator

Reminder

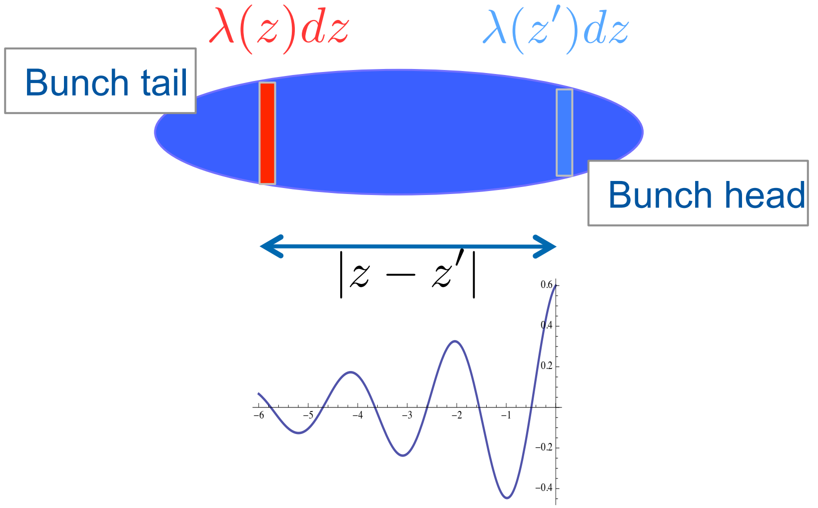


The wake terms have to be introduced in the equations of motion.

This allows the loop to be closed and we can study self-consistently the beam dynamics including the self-induced EM fields

Equations of longitudinal motion

- The single particle (or macroparticle) in the witness slice $\lambda(z)dz$ feels the force from
  - RF
  - The wake left from 'earlier' particles
  - The bunch own space charge
- The wake contribution can extend to several turns



$$\frac{dz}{dt} = -\eta c \delta$$

$$\frac{d\delta}{dt} = \frac{eV_{rf}(z)}{2\pi R p_0} - \frac{e^2}{2\pi R p_0} \sum_{k=0}^{\infty} \int_z^{\infty} \lambda(z' + 2\pi k R) W_{||}(z - z' - 2\pi k R) dz' - \frac{e^2 g \lambda'(z)}{2\pi \epsilon_0 \gamma^2 p_0}$$

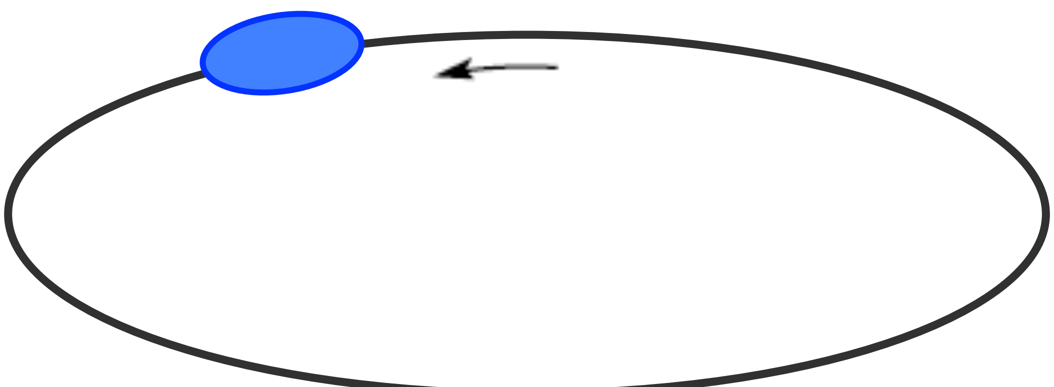
Equations of longitudinal motion

$$\frac{dz}{dt} = -\eta c \delta$$

$$\frac{d\delta}{dt} = \frac{eV_{rf}(z)}{2\pi R p_0} - \frac{e^2}{2\pi R p_0} \sum_{k=0}^{\infty} \int_z^{\infty} \lambda(z' + 2\pi k R) W_{||}(z - z' - 2\pi k R) dz' - \frac{e^2 g \lambda'(z)}{2\pi \epsilon_0 \gamma^2 p_0}$$

Interaction between bunch and (RF + wake) is assumed smeared all over the circumference

The wake is assumed to be the sum of the wakes from all elements along the ring (i.e. the longitudinal wake of the machine impedance model)



Wake force term
$F_{||}^{(W)}(z)$

Space charge term
→ Can also be associated to a wake function/impedance:

$$W_{||}^{(sc)}(z) = \frac{gR}{\epsilon_0 \gamma^2} \delta'(z)$$

$$\Downarrow$$

$$Z_{||}^{(sc)} = \frac{i\omega g R}{c^2 \epsilon_0 \gamma^2}$$

Equations of longitudinal motion

$$\frac{dz}{dt} = -\eta c \delta$$

$$\frac{d\delta}{dt} = \frac{eV_{rf}(z)}{2\pi R p_0} - \frac{e^2}{2\pi R p_0} \sum_{k=0}^{\infty} \int_z^{\infty} \lambda(z' + 2\pi k R) W_{||}(z - z' - 2\pi k R) dz' - \frac{e^2 g \lambda'(z)}{2\pi \epsilon_0 \gamma^2 p_0}$$

$$H = -\frac{1}{2}\eta c \delta^2 + \frac{eU(z)}{2\pi R p_0} + \frac{e^2}{2\pi R p_0} \int_{z_0}^{z} dz'' \sum_{k=0}^{\infty} \int_z^{\infty} \lambda(z' + 2\pi k R) W_{||}(z - z' - 2\pi k R) dz'$$

$$\psi_0(H) \text{ stationary solution of } \frac{\partial \psi}{\partial t} + \dot{z}\frac{\partial \psi}{\partial z} + \dot{\delta}\frac{\partial \psi}{\partial \delta} = 0$$

$$\psi_0(H) + \psi_1(z, \delta, t) \text{ also solution of } \frac{\partial \psi}{\partial t} + \dot{z}\frac{\partial \psi}{\partial z} + \dot{\delta}\frac{\partial \psi}{\partial \delta} = 0 \text{ with stable } \psi_1(z, \delta, t)$$

Regime of **potential well distortion** (i.e. perturbations to equilibrium solution are damped)

- Stable phase shift
- Synchrotron frequency shift
- Different matching (→ bunch lengthening for lepton machines)

Proportional to intensity

Equations of longitudinal motion

$$\frac{dz}{dt} = -\eta c \delta$$

$$\frac{d\delta}{dt} = \frac{eV_{rf}(z)}{2\pi R p_0} - \frac{e^2}{2\pi R p_0}\sum_{k=0}^{\infty}\int_z^{\infty}\lambda(z'+2\pi kR)W_{||}(z-z'-2\pi kR)dz' - \frac{e^2 g\lambda'(z)}{2\pi\epsilon_0\gamma^2 p_0}$$

$$H = -\frac{1}{2}\eta c\delta^2 + \frac{eU(z)}{2\pi R p_0} + \frac{e^2}{2\pi R p_0}\int_{z_0}^{z}dz''\sum_{k=0}^{\infty}\int_z^{\infty}\lambda(z'+2\pi kR)W_{||}(z-z'-2\pi kR)dz'$$

See Kevin's lecture on kinetic theory

$\psi_0(H)$ stationary solution of $\dot{z}\frac{\partial\psi}{\partial z} + \dot{\delta}\frac{\partial\psi}{\partial\delta} = 0$

$\psi_0(H) + \psi_1(z,\delta,t)$ solution of $\frac{\partial\psi}{\partial t} + \dot{z}\frac{\partial\psi}{\partial z} + \dot{\delta}\frac{\partial\psi}{\partial\delta} = 0$ with growing $\psi_1(z,\delta,t)$

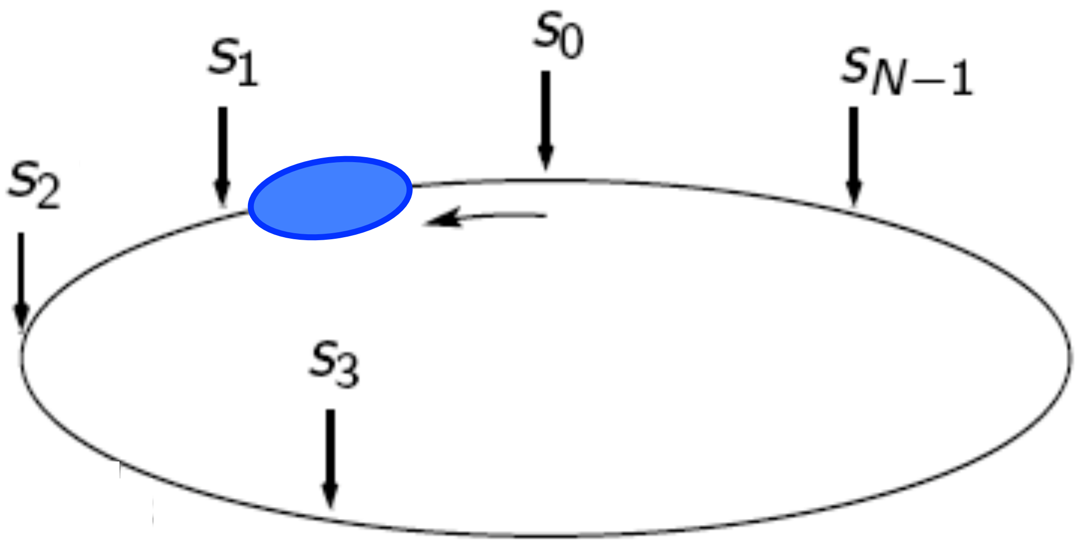
Regime of **longitudinal instability** (i.e. perturbations to equilibrium solution grow exponentially)

- Dipole mode instabilities
- Coupled bunch instabilities
- Microwave instability (longitudinal mode coupling)

Numerical implementation (longitudinal)

$$z_{n+1} = z_n - 2\pi R\eta\delta_n$$

$$\delta_{n+1} = \delta_n + \frac{eV_{rf}(z_{n+1})}{cp_0} - \frac{e^2}{cp_0}\int_{-\infty}^{\infty}\lambda_{n+1}(z')W_{||}(z_{n+1}-z')dz' - \frac{e^2 gRZ_0\lambda'_{n+1}(z_{n+1})}{\gamma^2 p_0}$$



We assume that interaction of bunch with (RF + wake + space charge) lumped at one or more locations.

This is usually true for RF, it is a one-kick approximation for space charge and wake

Wake from impedance model (can be divided by number of points, if more than one)

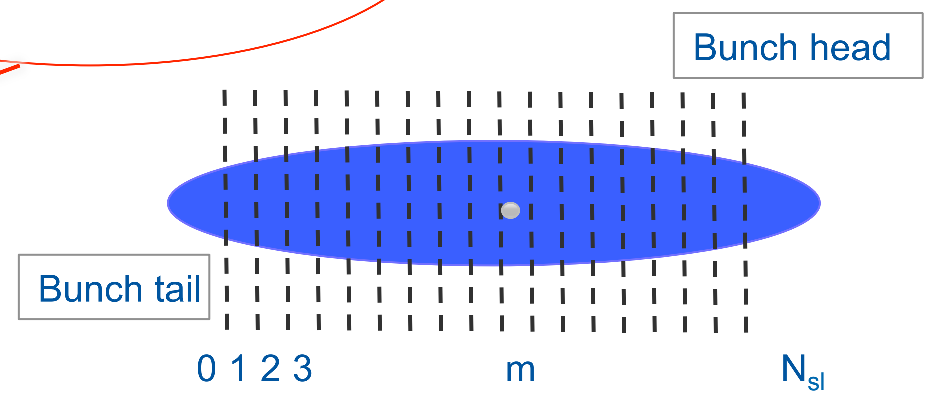
Numerical implementation (longitudinal)

$$z_{n+1} = z_n - 2\pi R\eta\delta_n$$

Single turn wake

$$\delta_{n+1} = \delta_n + \frac{eV_{rf}(z_{n+1})}{cp_0} - \frac{e^2}{cp_0}\int_{-\infty}^{\infty}\lambda_{n+1}(z')W_{||}(z_{n+1}-z')dz' - \frac{e^2 gRZ_0\lambda'_{n+1}(z_{n+1})}{\gamma^2 p_0}$$

$$\sum_{h=m}^{N_{sl}}\lambda_{n+1}(h\Delta z_{n+1})W_{||}[(m-h)\Delta z_{n+1}]$$



- ***N_{sl}*** is the number of slices in which a bunch is subdivided
- ***m*** is the slice index where ***z_{n+1}*** is located
- ***Δz_{n+1}*** is the slice width at step ***n+1***
- ***Slicing*** needs to be fine enough as to sample the wake function
- Slice ***m*** included because of ***energy loss***
- ***Discontinuity in z=0*** make slicing requirement more stringent (need check of convergence of energy loss value)

Number of particles in slice h

$$\lambda_{n+1}(h\Delta z_{n+1}) = \frac{\Delta N_h}{\Delta z_{n+1}}$$

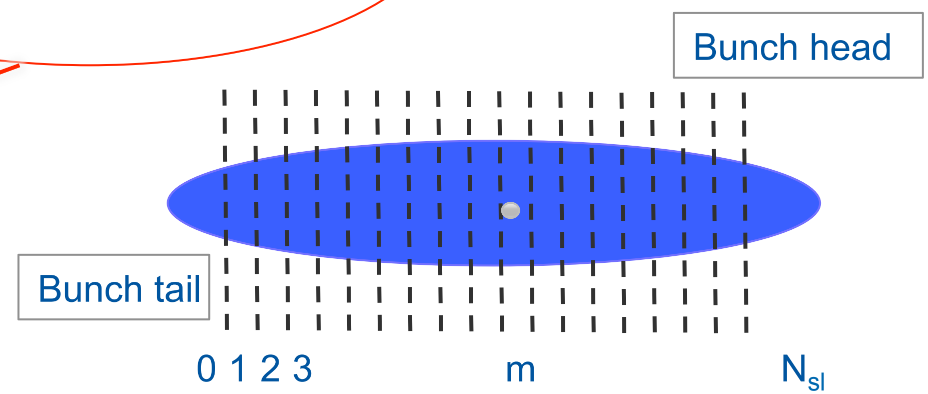
Numerical implementation (longitudinal)

$$z_{n+1} = z_n - 2\pi R\eta\delta_n$$

Single turn wake

$$\delta_{n+1} = \delta_n + \frac{eV_{rf}(z_{n+1})}{cp_0} - \frac{e^2}{cp_0}\int_{-\infty}^{\infty}\lambda_{n+1}(z')W_{||}(z_{n+1}-z')dz' - \frac{e^2 gRZ_0\lambda'_{n+1}(z_{n+1})}{\gamma^2 p_0}$$

$$\sum_{h=m}^{N_{sl}} \lambda_{n+1}(h\Delta z_{n+1})W_{||}[(m-h)\Delta z_{n+1}]$$



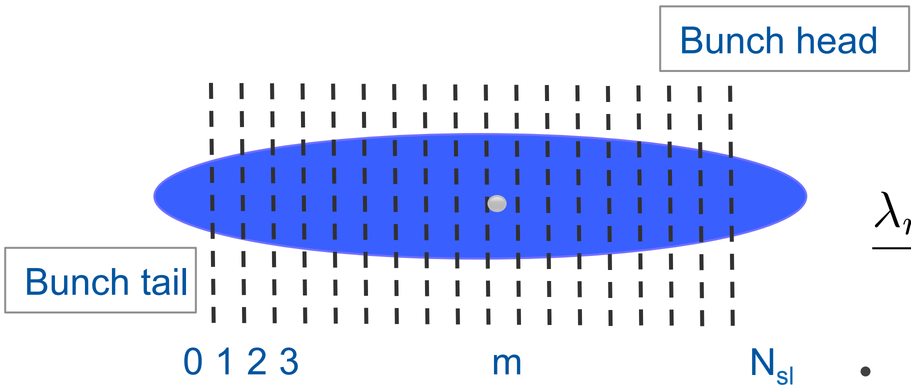
For multi-turn wakes (i.e. preserving memory of the wake over $\mathbf{n_t}$ turns)

$$\sum_{k=0}^{n_t}\sum_{h=0}^{N_{sl}} \lambda_{n+1-k}(h\Delta z_{n+1-k})W_{||}[(m-h)\Delta z_{n+1-k} - 2\pi kR - \langle z\rangle_{n+1-k} + \langle z\rangle_{n+1}]$$

Numerical implementation (longitudinal)

$$z_{n+1} = z_n - 2\pi R \eta \delta_n$$

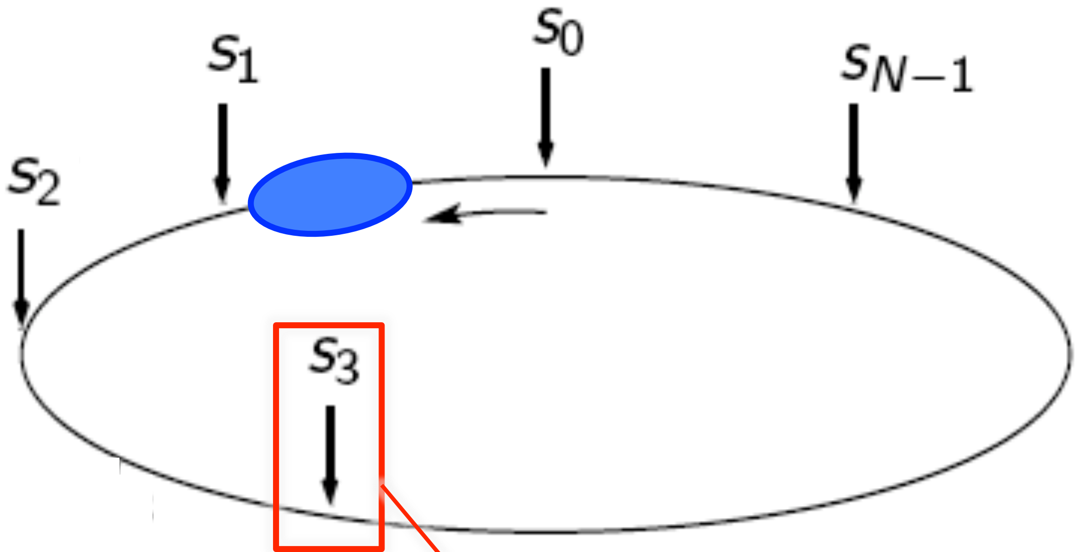
$$\delta_{n+1} = \delta_n + \frac{eV_{rf}(z_{n+1})}{cp_0} - \frac{e^2}{cp_0}\int_{-\infty}^{\infty} \lambda_{n+1}(z')W_{||}(z_{n+1}-z')dz' - \frac{e^2 gRZ_0\lambda'_{n+1}(z_{n+1})}{\gamma^2 p_0}$$



$$\frac{\lambda_{n+1}[(m+1)\Delta z_{n+1}] - \lambda_{n+1}[(m-1)\Delta z_{n+1}]}{2\Delta z_{n+1}}$$

- Too fine ***slicing*** (i.e. too low number of macroparticles per slice) can be the origin of noise problems
  - → If there are no other constraints in the simulation, slicing can be chosen such as to have a smooth derivative
  - → ***Smoothing*** can be applied to both distribution before differentiating and to derivative
- Alternatively, space charge force can be directly calculated from E_s with **Poisson solver** (3 or 2.5D)

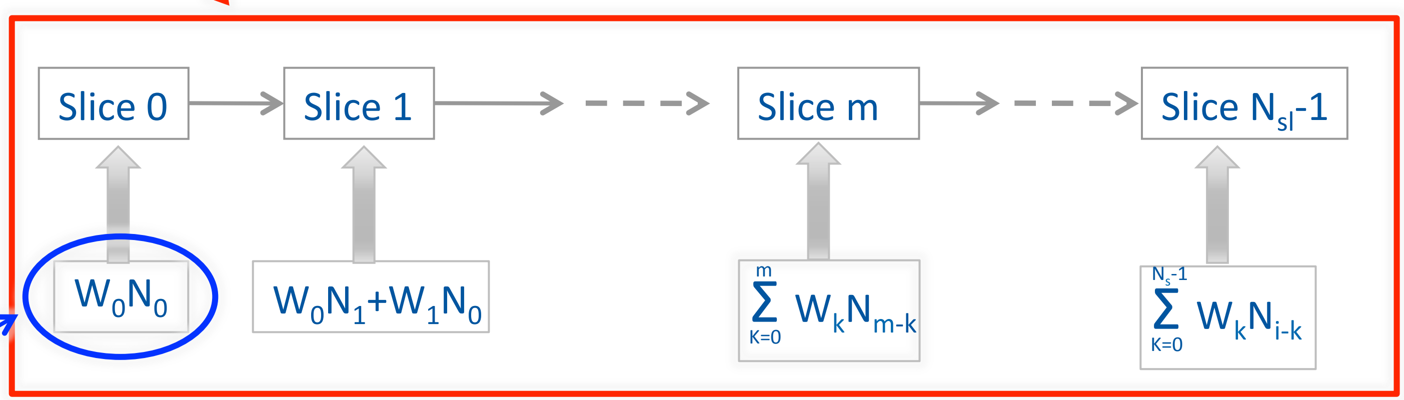
Numerical implementation (longitudinal)



At each interaction point macroparticles in each slice receive the kick from the wakes of the preceding slices.
Slicing is refreshed at each turn taking into account the longitudinal motion

Longitudinal wake

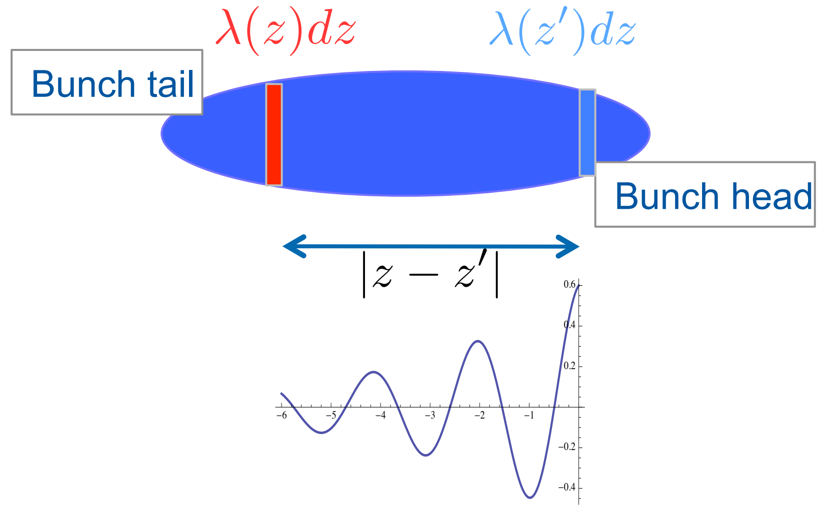
$W_m = W_{||}(m\ \Delta z)$



Energy loss

Equations of transverse motion

- The single particle (or macroparticle) in the witness slice λ(z)dz feels the force from
  - External focusing
  - The wake left from 'earlier' particles
  - The bunch own space charge



$$\frac{dx}{ds} = x'$$

$$\frac{dx'}{ds} + \left(\frac{Q_{x0}}{R}\right)^2 x = 0$$

$$= -\frac{e^2}{2\pi R E_0} \sum_{k=0}^{\infty} \int_{-\infty}^{\infty} \lambda(z' + 2k\pi R)[W_x(z - z' - 2k\pi R)\langle x\rangle(z' + 2k\pi R) +$$

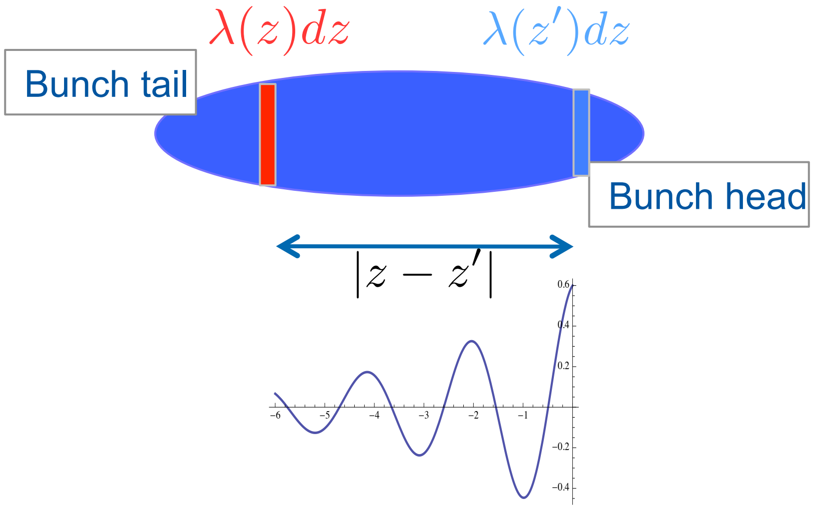
$$+ W_{Qx}(z - z' - 2k\pi R)x]dz' +$$

$$+\lambda(z) \cdot \mathcal{F}\left(x - \langle x\rangle(z)\right)$$

Nonlinear force depending on the local density

Equations of transverse motion

- The single particle (or macroparticle) in the witness slice λ(z)dz feels the force from
  - External focusing
  - The wake left from ‘earlier’ particles
  - The bunch own space charge



$$\frac{dx}{ds} = x'$$

$$\frac{dx'}{ds} + \left(\frac{Q_{x0}}{R}\right)^2 x =$$

$$= -\frac{e^2}{2\pi R E_0} \sum_{k=0}^{\infty} \int_{-\infty}^{\infty} \lambda(z' + 2k\pi R)[W_x(z - z' - 2k\pi R)\langle x\rangle(z' + 2k\pi R) +$$

$$+ W_{Qx}(z - z' - 2k\pi R)x]dz' +$$

$$+ \frac{e^2\lambda(z)\left[(x - \langle x\rangle(z)\right]}{\epsilon_0 E_0 \langle\sigma_x\rangle(z)[\langle\sigma_x\rangle(z) + \langle\sigma_y\rangle(z)]}$$

A simplified linearized expression from the Bassetti-Erskine formula for Gaussian transverse distribution

Equations of transverse motion

$$\frac{dx}{ds} = x'$$

$$\frac{dx'}{ds} + \left(\frac{Q_{x0}}{R}\right)^2 x = -\frac{e^2}{2\pi R E_0}\int_{-\infty}^{\infty} \lambda(z')\left[W_x(z-z')\langle x\rangle(z') + W_{Qx}(z-z')x\right] dz'$$

N.B. From now on, here we will omit the summation over the previous turns and space charge term

$$\frac{dz}{ds} = -\eta\delta$$

$$\frac{d\delta}{ds} = \frac{eV_{rf}(z)}{2\pi R E_0}$$

Need to couple to equations of longitudinal motion because of z-dependent wake
→ Neglect longitudinal wake

$$H = \frac{1}{2}\frac{Q_{x0}}{R}\left[\left(\frac{Q_{x0}}{R}\right)x^2 + \left(\frac{R}{Q_{x0}}\right)x'^2\right] + \frac{eU(z)}{2\pi R E_0} - \frac{\eta}{2}\delta^2 +$$
$$+\frac{e^2}{2\pi R E_0}\left[x\int_z^{\infty}\lambda(z')\langle x\rangle(z')W_x(z-z')dz' + x^2\int_z^{\infty}\lambda(z')W_{Qx}(z-z')dz'\right]$$

Equations of transverse motion

Change of variables in the transverse phase plane
x, x' → J_x, θ_x

$$\begin{cases} x = \sqrt{2J_x \dfrac{R}{Q_{x0}}} \cos\theta_x \\ x' = \sqrt{2J_x \dfrac{Q_{x0}}{R}} \sin\theta_x \end{cases}$$

$$\psi_0(H) \text{ stationary solution of } \frac{\partial\psi}{\partial t} + z'\frac{\partial\psi}{\partial z} + \delta'\frac{\partial\psi}{\partial\delta} + J_x'\frac{\partial\psi}{\partial J_x} + \theta_x'\frac{\partial\psi}{\partial\theta_x} = 0$$

$$\psi_0(H) + \psi_1(J_x, \theta_x, z, \delta, t) \text{ also solution with stable } \psi_1$$

Beam transversely stable (i.e. perturbations to equilibrium solution are damped)

- Head-tail modes
- Coherent betatron tune shift with intensity

Equations of transverse motion

$$\psi_0(H)\ \text{stationary solution of}\ \frac{\partial \psi}{\partial t} + z'\frac{\partial \psi}{\partial z} + \delta'\frac{\partial \psi}{\partial \delta} + J'_x\frac{\partial \psi}{\partial J_x} + \theta'_x\frac{\partial \psi}{\partial \theta_x} = 0$$

$$\psi_0(H) + \psi_1(J_x, \theta_x, z, \delta, t)\ \text{also solution with}\ \text{growing}\ \psi_1$$

Beam transversely unstable (i.e. perturbations to eq[illegible]ution grow)

- Head-tail modes with non-zero chromaticity [illegible]ping
- Transverse mode coupling instability [illegible] a current threshold

See Kevin's lecture on kinetic theory

Several **codes dev**[illegible]**olve Vlasov equation** and determine unstable (most critical[illegible] well as stability regions. Some can include multi-bunch, chromaticity[illegible]itude detuning and transverse damper, but they all consider linearised longitudinal motion and only dipole wake fields. E.g.

- MOSES [Y. Chin, CERN/SPS/85-2 & CERN/LEP-TH/88-05]
- NHTVS [A. Burov, Phys. Rev. ST AB 17, 021007 (2014)]
- DELPHI [N. Mounet, HSC Meeting, 09/04/2014]

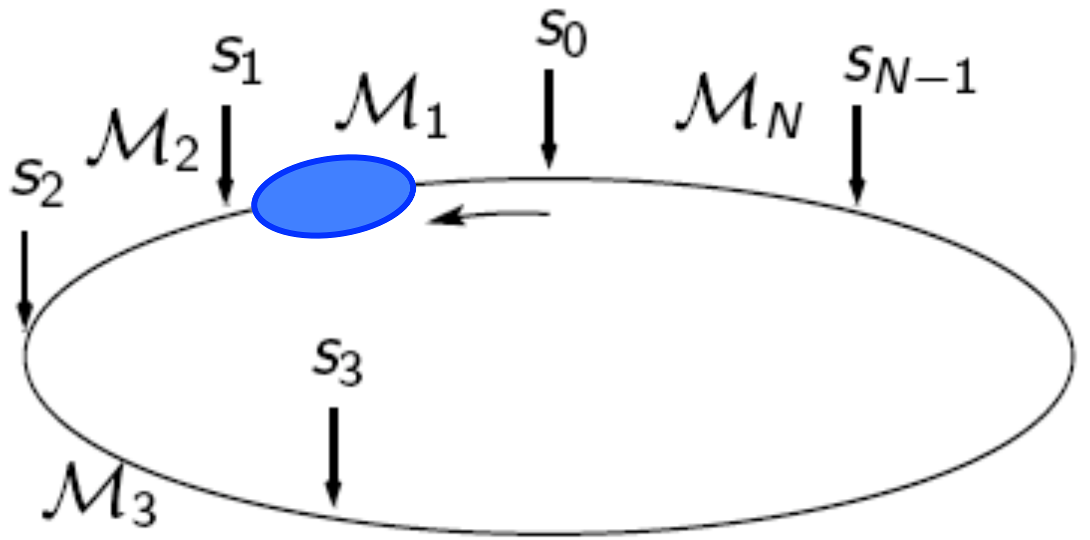
Numerical implementation (transverse)

$$\begin{pmatrix} x_{n+1} \\ x'_{n+1} \end{pmatrix} = \underline{\underline{\mathcal{M}}}(\delta, J_x) \cdot \begin{pmatrix} x_n \\ x'_n - \dfrac{e^2}{2\pi R E_0} \displaystyle\int_{-\infty}^{\infty} \lambda_n(z') \left[W_x(z-z') \langle x \rangle_n(z') + W_{Qx}(z-z') x_n \right] dz' \end{pmatrix}$$

$$z_{n+1} = z_n - 2\pi R \eta \delta_n$$

$$\delta_{n+1} = \delta_n + \frac{e V_{rf}(z_{n+1})}{E_0}$$

Assuming interaction of bunch with wake lumped at one or more locations + 1-turn (or sector) map transport



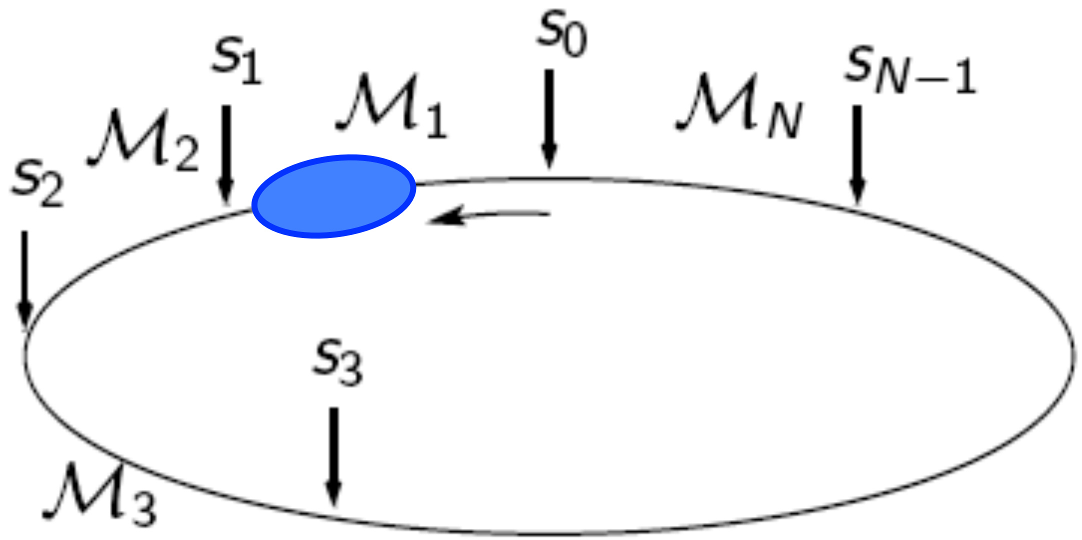
Numerical implementation (transverse)

$$\begin{pmatrix} x_{n+1} \\ x'_{n+1} \end{pmatrix} = \underline{\underline{\mathcal{M}}}(\delta, J_x) \cdot \begin{pmatrix} x_n \\ x'_n - \dfrac{e^2}{2\pi R E_0} \displaystyle\int_{-\infty}^{\infty} \lambda_n(z') \left[W_x(z - z') \langle x \rangle_n (z') + W_{Qx}(z - z') x_n \right] dz' \end{pmatrix}$$

$$z_{n+1} = z_n - 2\pi R \eta \delta_n$$

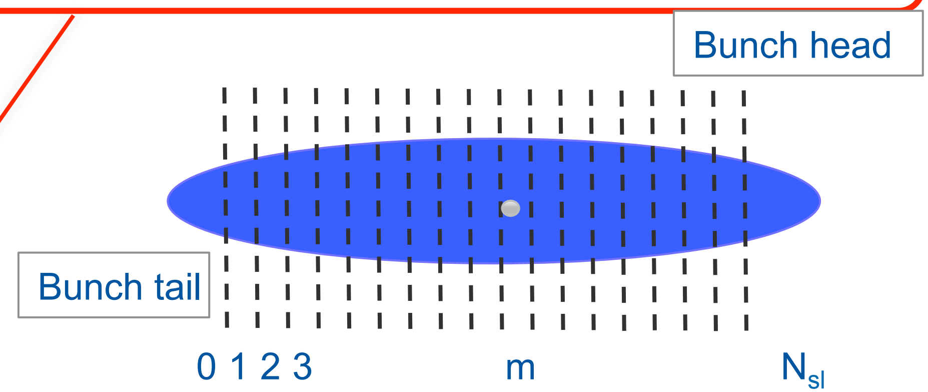
$$\delta_{n+1} = \delta_n + \frac{e V_{rf}(z_{n+1})}{E_0}$$

- Usually, the 1-turn transport matrix is built from the average beta function along the ring $\langle\beta_x\rangle = R/Q_{x0}$ and the tune $Q_x(\delta, J_x)$
- The wakes W_x and W_{Qx} are those from the impedance model (divided by the number of points, if more than one)



Numerical implementation (transverse)

$$\begin{pmatrix} x_{n+1} \\ x'_{n+1} \end{pmatrix} = \underline{\underline{\mathcal{M}}}(\delta, J_x) \cdot \begin{pmatrix} x_n \\ x'_n - \dfrac{e^2}{2\pi R E_0} \boxed{\int_{-\infty}^{\infty} \lambda_n(z') \left[W_x(z-z') \langle x \rangle_n (z') + W_{Qx}(z-z') x_n \right] dz'} \end{pmatrix}$$

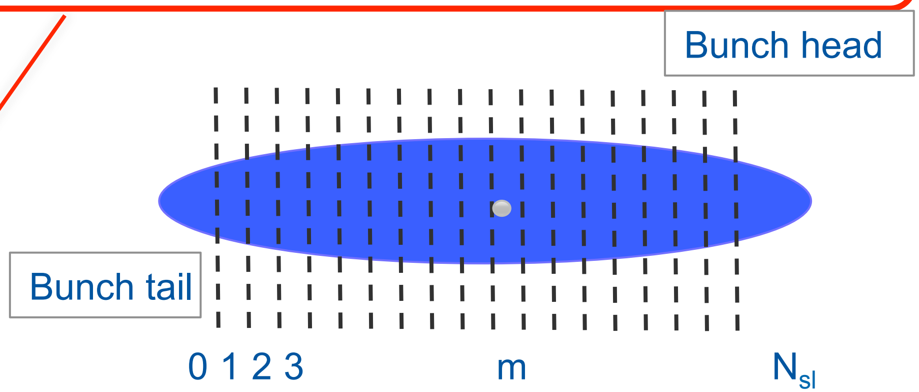


$$\sum_{h=m+1}^{N_{sl}} \lambda_n(h\Delta z_n) \left(W_x[(m-h)\Delta z_n] \langle x \rangle_n (h\Delta z_n) + W_{Qx}[(m-h)\Delta z_n] x_n \right)$$

- ***N$_{sl}$*** is the number of slices in which a bunch is subdivided
- ***m*** is the slice index where ***z$_{n+1}$*** is located
- ***Δz$_{n+1}$*** is the slice width at step ***n+1***
- ***Slicing*** needs to be fine enough as to sample the wake function
- If ***indirect space charge included***, sum runs from *0* to *N$_{sl}$* and peak on source slice needs to be resolved correctly

Numerical implementation (transverse)

$$\begin{pmatrix} x_{n+1} \\ x'_{n+1} \end{pmatrix} = \underline{\underline{\mathcal{M}}}(\delta, J_x) \cdot \begin{pmatrix} x_n \\ x'_n - \frac{e^2}{2\pi R E_0} \boxed{\int_{-\infty}^{\infty} \lambda_n(z') \left[W_x(z-z') \langle x \rangle_n (z') + W_{Qx}(z-z') x_n \right] dz'} \end{pmatrix}$$



$$\sum_{h=m+1}^{N_{sl}} \lambda_n(h\Delta z_n) \left(W_x[(m-h)\Delta z_n] \langle x \rangle_n (h\Delta z_n) + W_{Qx}[(m-h)\Delta z_n] x_n \right)$$

For multi-turn wakes
(i.e. preserving memory of the wake over $\mathbf{n_t}$ turns)

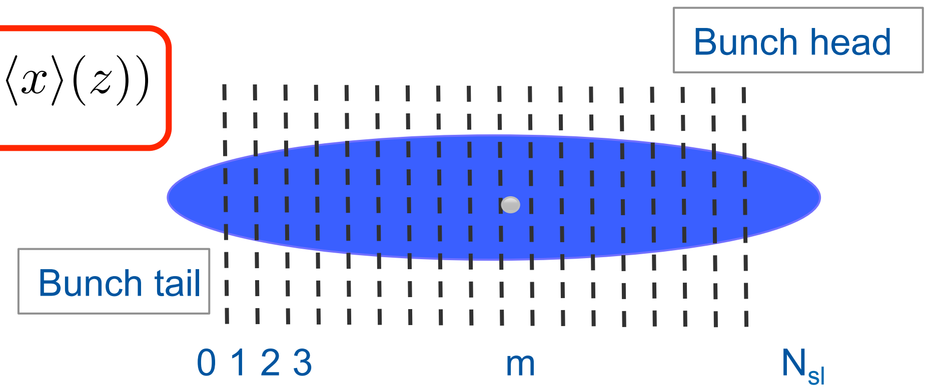
$$\sum_{k=0}^{n_t} \sum_{h=m+1}^{N_{sl}} \lambda_{n-k}(h\Delta z_{n-k}) \left(W_x[(m-h)\Delta z_{n-k} - 2\pi k R - \langle z \rangle_{n-k} + \langle z \rangle_n] \langle x \rangle_{n-k} (h\Delta z_{n-k}) + \right.$$

$$\left. + W_{Qx}[(m-h)\Delta z_{n-k} - 2\pi k R - \langle z \rangle_{n-k} + \langle z \rangle_n] x_n \right)$$

Numerical implementation (transverse)

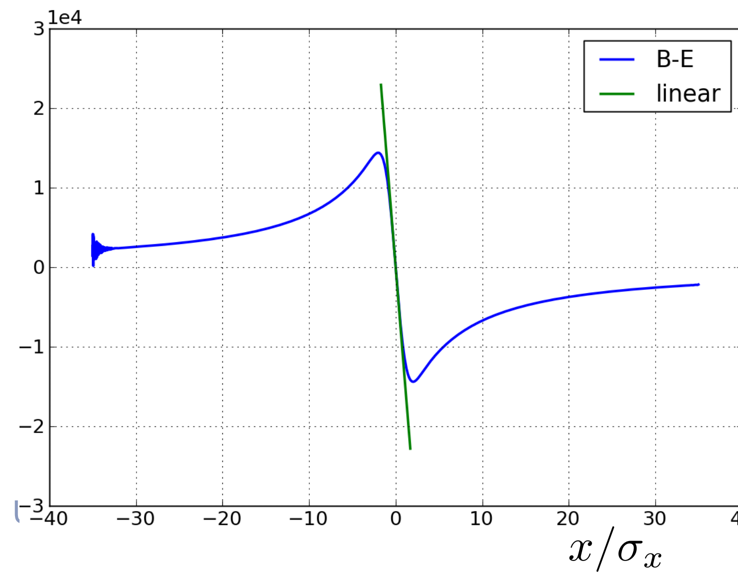
$$\begin{pmatrix} x_{n+1} \\ x'_{n+1} \end{pmatrix} = \underline{\underline{\mathcal{M}}}(\delta, J_x) \cdot \begin{pmatrix} x_n \\ x'_n - \dfrac{e^2}{2\pi R E_0} \displaystyle\int_{-\infty}^{\infty} \lambda_n(z') \left[W_x(z-z')\langle x\rangle_n(z') + W_{Qx}(z-z') x_n \right] dz' \end{pmatrix}$$

$$\boxed{+\lambda_n(z) \cdot \mathcal{F}\left(x - \langle x\rangle(z)\right)}$$



$$\lambda_n(m\Delta z_n) \cdot \mathcal{F}_n\left(x - \langle x\rangle_n(m\Delta z_n), \sigma_{(x,y)n}(m\Delta z_n)\right)$$

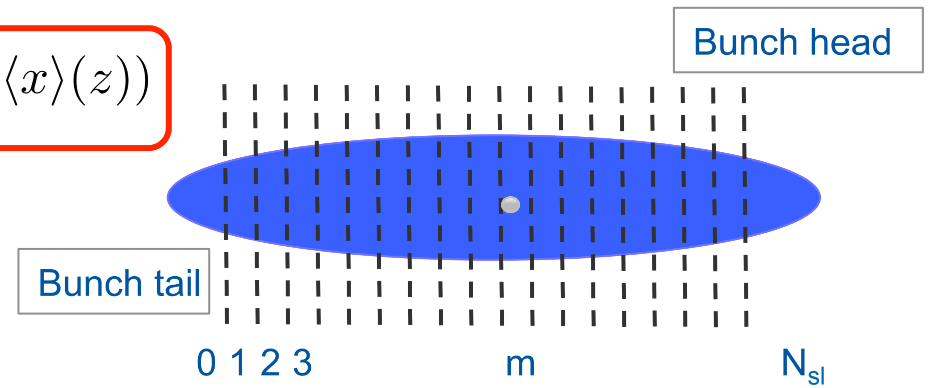
Space charge force from m[th] slice can be calculated with Bassetti-Erskine formula (soft-Gaussian approximation)



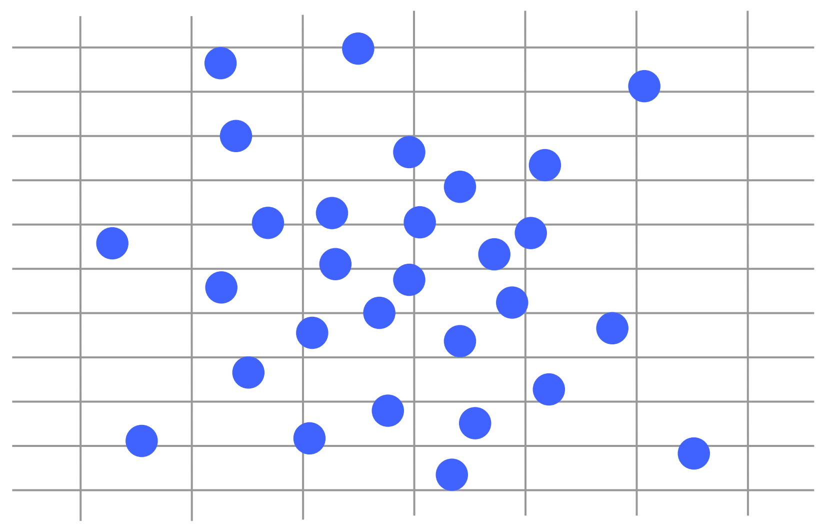
Numerical implementation (transverse)

$$\begin{pmatrix} x_{n+1} \\ x'_{n+1} \end{pmatrix} = \underline{\underline{\mathcal{M}}}(\delta, J_x) \cdot \left(\begin{array}{c} x_n \\ x'_n - \dfrac{e^2}{2\pi R E_0} \displaystyle\int_{-\infty}^{\infty} \lambda_n(z') \left[W_x(z-z') \langle x \rangle_n(z') + W_{Qx}(z-z') x_n \right] dz' \end{array} \right)$$

$$\boxed{+\lambda_n(z) \cdot \mathcal{F}\left(x - \langle x \rangle(z)\right)}$$

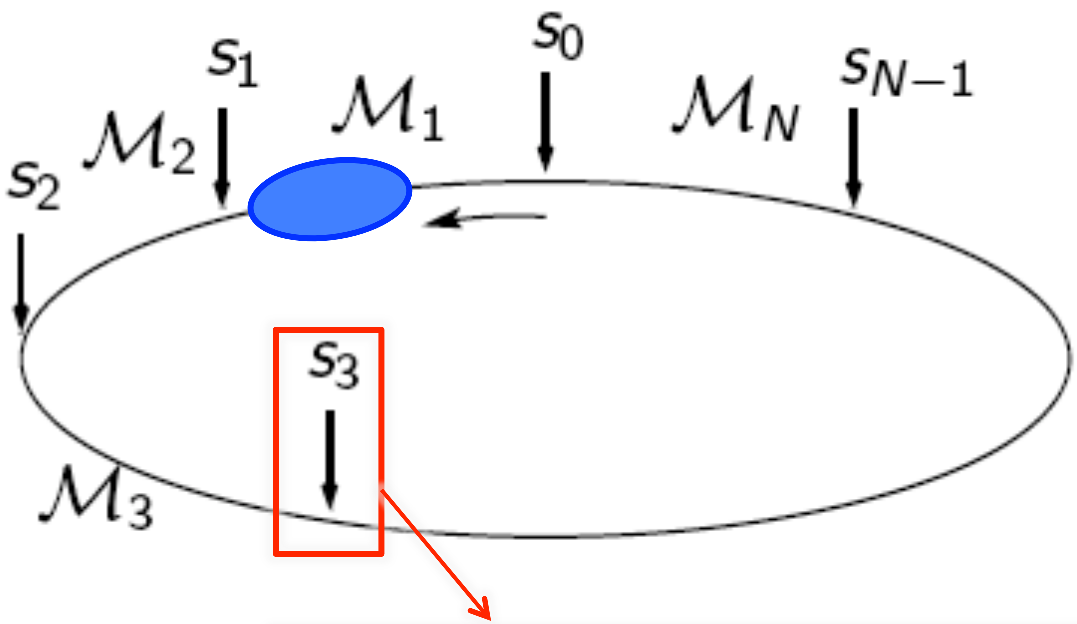


$$\lambda_n(m\Delta z_n) \cdot \mathcal{F}_n \left(x - \langle x \rangle_n(m\Delta z_n), \sigma_{(x,y)n}(m\Delta z_n)\right)$$



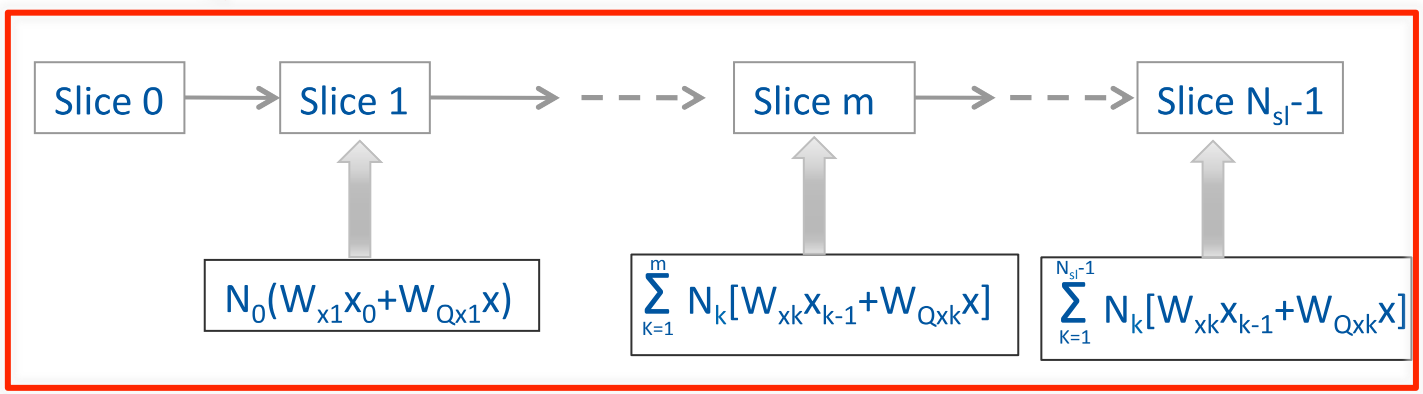
Space charge force from m^{th} slice can be calculated with 2D PIC solver from the macroparticle distribution in the slice m (PyHEADTAIL exercise from yesterday!)

Numerical implementation (transverse)



At each interaction point macroparticles in each slice receive the kick from the wakes of the preceding slices.
Slicing is refreshed at each turn taking into account the longitudinal motion

Transverse (x)



Dipolar:
$W_{xm} = W_x(m\,\Delta z)$
Quadrupolar:
$W_{Qxm} = W_{Qx}(m\,\Delta z)$
x_m centroid of slice m
x position of particle

One-particle models

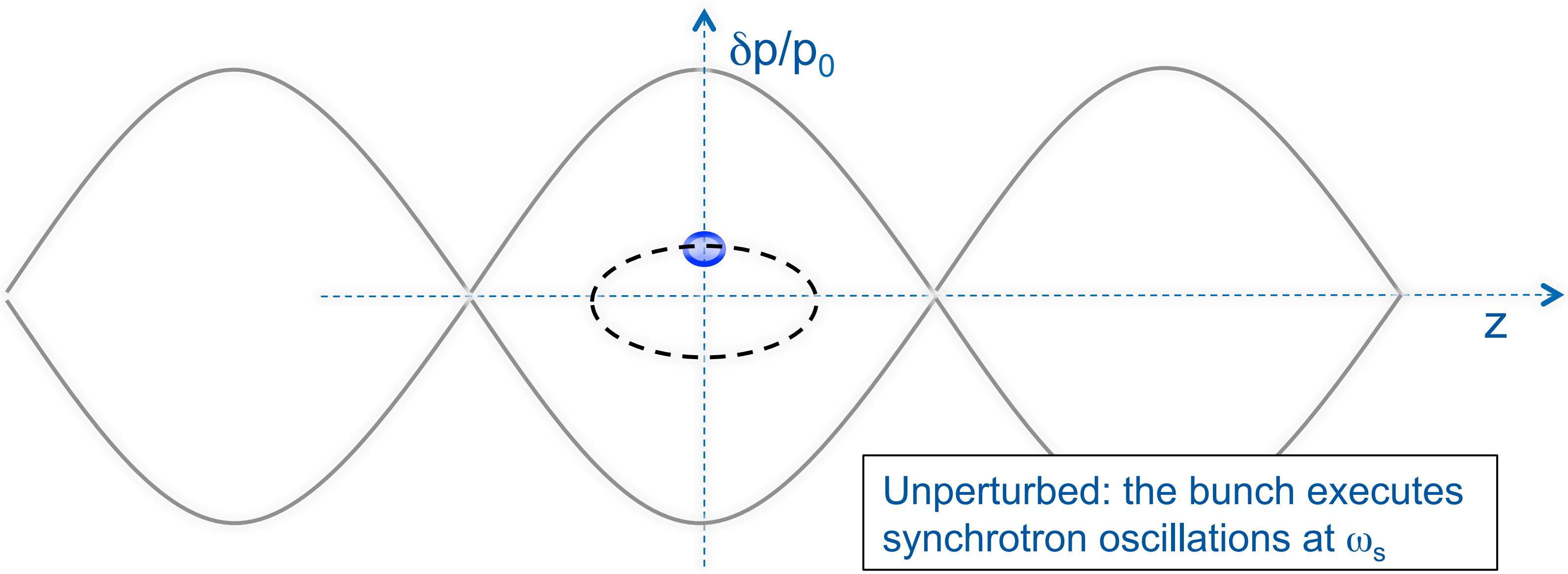
- Longitudinal plane → Robinson instability
- Transverse plane → Rigid dipole instability

One-particle models

- ❑ **Longitudinal plane → Robinson instability**
- ❑ **Transverse plane → Rigid dipole instability**

The Robinson instability

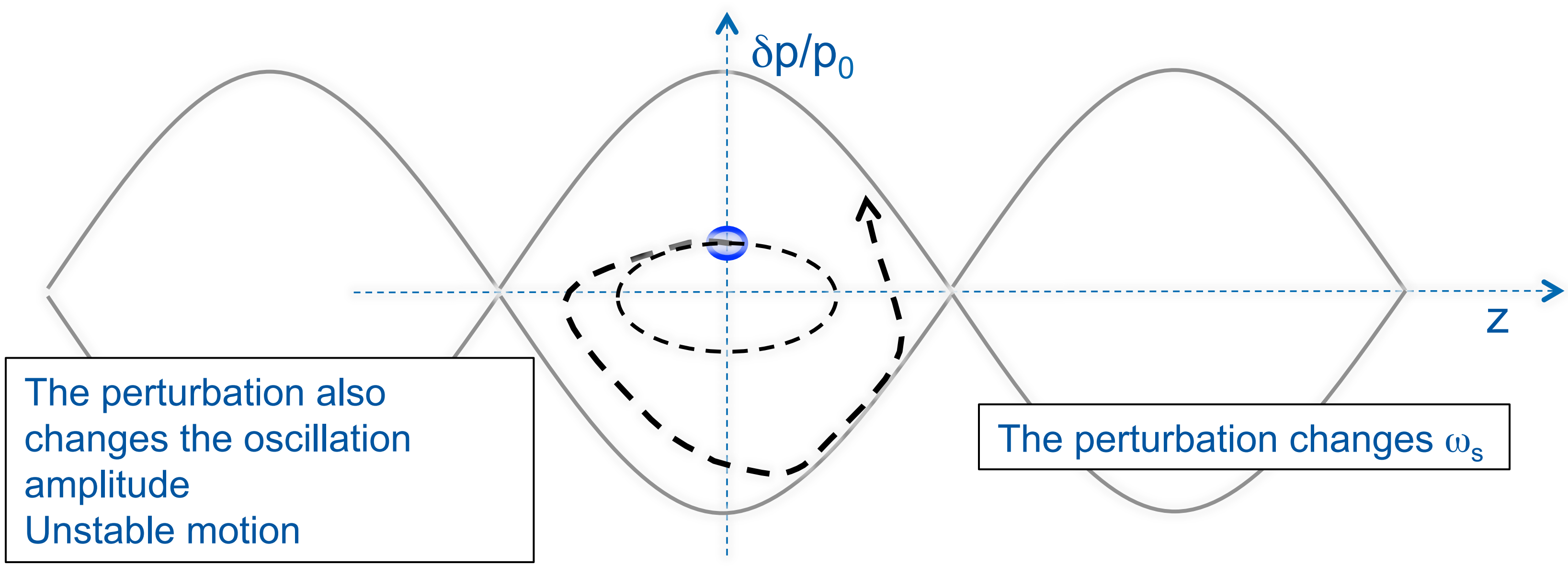
- To illustrate the Robinson instability we will use some simplifications:
  - ⇒ The bunch is point-like and feels an external linear force (i.e. it would execute linear synchrotron oscillations in absence of the wake forces)
  - ⇒ The bunch additionally feels the effect of a multi-turn wake



Unperturbed: the bunch executes synchrotron oscillations at ω_s

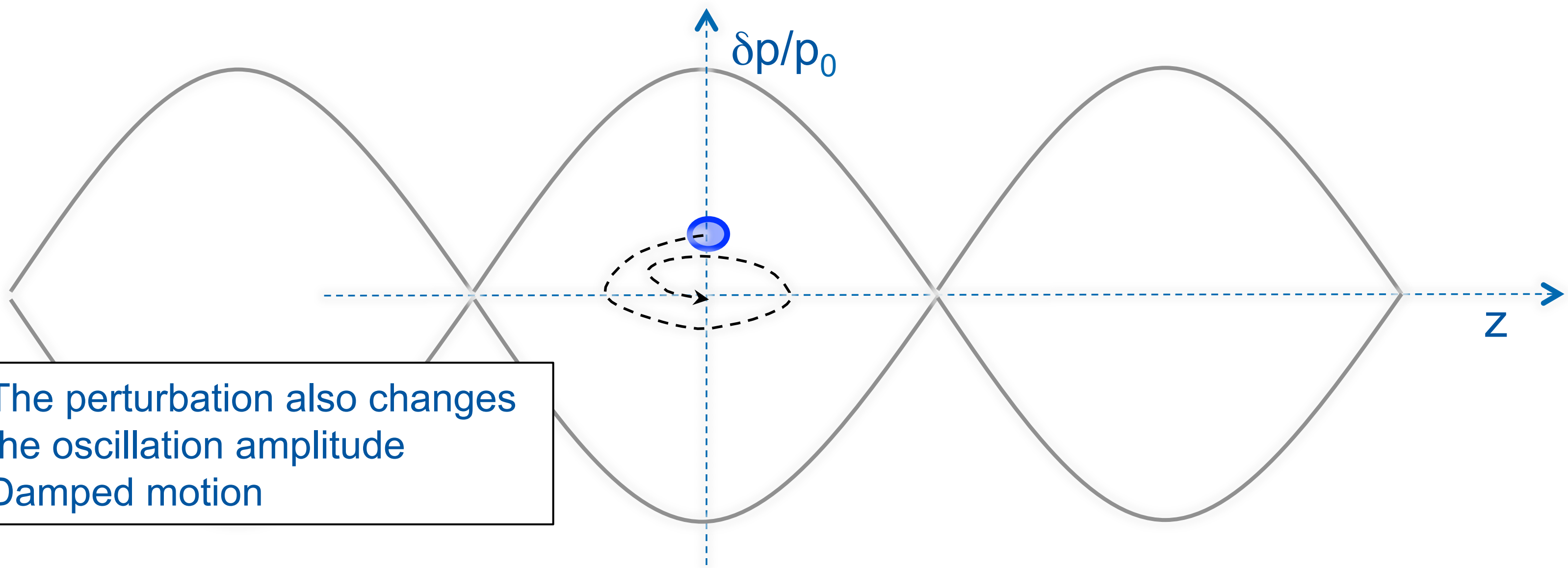
The Robinson instability

– To illustrate the Robinson instability we will use some simplifications:
  - ⇒ The bunch is point-like and feels an external linear force (i.e. it would execute linear synchrotron oscillations in absence of the wake forces)
  - ⇒ The bunch additionally feels the effect of a multi-turn wake



The Robinson instability

– To illustrate the Robinson instability we will use some simplifications:
  - ⇒ The bunch is point-like and feels an external linear force (i.e. it would execute linear synchrotron oscillations in absence of the wake forces)
  - ⇒ The bunch additionally feels the effect of a multi-turn wake



The perturbation also changes the oscillation amplitude
Damped motion

The Robinson instability

- To illustrate the Robinson instability we will use some simplifications:
  - ⇒ The bunch is point-like and feels an external linear force (i.e. it would execute linear synchrotron oscillations in absence of the wake forces)
  - ⇒ The bunch additionally feels the effect of a multi-turn wake

$$\frac{d^2 z}{dt^2} + \frac{\eta e V_{\mathrm{rf}}(z)}{m_0 \gamma C} = \frac{\eta e^2}{m_0 \gamma C} \sum_{k=-\infty}^{\infty} \int_{-\infty}^{\infty} \lambda(z' + kC) W_{||}(z - z' - kC) dz'$$

External RF

Wake fields

$$\frac{d^2 z}{dt^2} + \omega_s^2 z = \frac{N e^2 \eta}{C m_0 \gamma} \sum_{k=-\infty}^{\infty} W_{||} \left[z(t) - z(t - kT_0) - kC \right]$$

The Robinson instability

- To illustrate the Robinson instability we will use some simplifications:
  - ⇒ The bunch is point-like and feels an external linear force (i.e. it would execute linear synchrotron oscillations in absence of the wake forces)
  - ⇒ The bunch additionally feels the effect of a multi-turn wake

$$\frac{d^2 z}{dt^2} + \omega_s^2 z = \frac{Ne^2\eta}{Cm_0\gamma} \sum_{k=-\infty}^{\infty} W_{||}\left[z(t) - z(t - kT_0) - kC\right]$$

We assume that the wake can be linearized on the scale of the oscillation amplitude

$$W_{||}\left[z(t) - z(t - kT_0) - kC\right] \approx W_{||}(kC) + W_{||}'(kC) \cdot \left[z(t) - z(t - kT_0)\right]$$

The Robinson instability

$$W_{||}\left[z(t)-z(t-kT_0)-kC\right] \approx \cancel{W_{||}(kC)} + W'_{||}(kC)\cdot\left[z(t)-z(t-kT_0)\right]$$

- ⇒ The term $\Sigma\, W_{||}(kC)$ only contributes to a constant term in the solution of the equation of motion, i.e. the synchrotron oscillation will be executed around a certain z_0 and not around 0. This term represents the stable phase shift that compensates for the energy loss
- ⇒ The dynamic term proportional to $z(t)-z(t-kT_0) \approx kT_0 dz/dt$ will introduce a "friction" term in the equation of the oscillator, which can lead to instability!

$$z(t) \propto \exp\left(-i\Omega t\right)$$

$$\Omega^2-\omega_s^2 = -\frac{Ne^2\eta}{Cm_0\gamma}\boxed{\sum_{k=-\infty}^{\infty}\left[1-\exp\left(-ik\Omega T_0\right)\right]\cdot W'_{||}(kC)}$$

$$\boxed{i\cdot\frac{1}{C}\sum_{p=-\infty}^{\infty}\left[p\omega_0 Z_{||}(p\omega_0)-(p\omega_0+\Omega)Z_{||}(p\omega_0+\Omega)\right]}$$

The Robinson instability

⇒ We assume a small deviation from the synchrotron tune
⇒ Re(Ω - ω_s) → Synchrotron tune shift
⇒ Im(Ω - ω_s) → Growth/damping rate, only depends on the dynamic term, if it is positive there is an instability!

$$\Omega^2 - \omega_s^2 \approx 2\omega_s \boxed{(\Omega - \omega_s)}$$

Complex frequency shift

$$\Delta\omega_s = \mathrm{Re}(\Omega - \omega_s) = \left(\frac{e^2}{m_0 c^2}\right) \frac{N\eta}{2\gamma T_0^2 \omega_s} \text{ x}$$

$$\text{x} \sum_{p=-\infty}^{\infty} \left[p\omega_0 \mathrm{Im} Z_{||}(p\omega_0) - (p\omega_0 + \omega_s)\mathrm{Im} Z_{||}(p\omega_0 + \omega_s)\right]$$

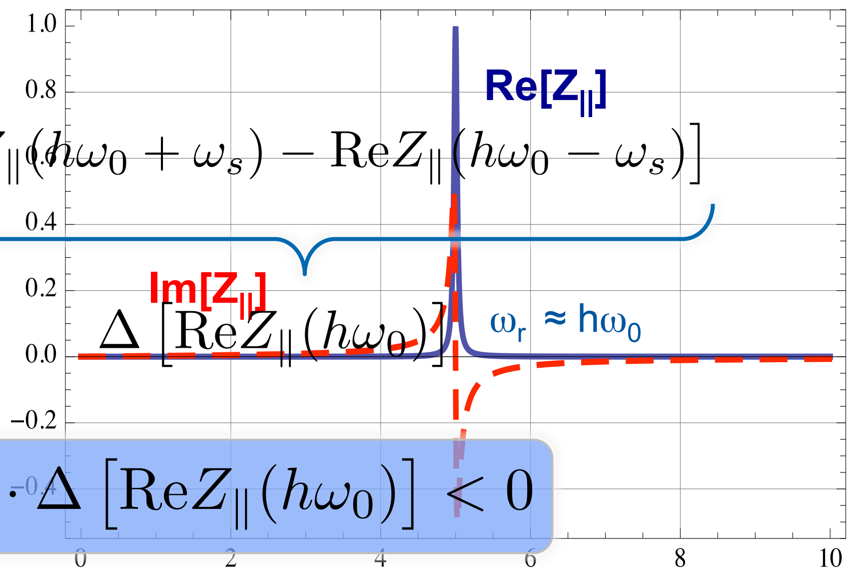
$$\tau^{-1} = \mathrm{Im}(\Omega - \omega_s) = \left(\frac{e^2}{m_0 c^2}\right) \frac{N\eta}{2\gamma T_0^2 \omega_s} \sum_{p=-\infty}^{\infty} (p\omega_0 + \omega_s)\mathrm{Re} Z_{||}(p\omega_0 + \omega_s)$$

The Robinson instability

$$\tau^{-1} = \mathrm{Im}(\Omega - \omega_s) = \left(\frac{e^2}{m_0 c^2}\right) \frac{N\eta}{2\gamma T_0^2 \omega_s} \sum_{p=-\infty}^{\infty} (p\omega_0 + \omega_s) \mathrm{Re} Z_{\parallel}(p\omega_0 + \omega_s)$$

- ⇒ We assume the impedance to be peaked at a frequency ω_r close to $h\omega_0 >> \omega_s$ (e.g. RF cavity fundamental mode or HOM)
- ⇒ Only two dominant terms are left in the summation at the RHS of the equation for the growth rate
- ⇒ Stability requires that η and $\Delta[\mathrm{Re}\, Z_{\parallel}(h\omega_0)]$ have different signs

$$\tau^{-1} = \left(\frac{e^2}{m_0 c^2}\right) \frac{N\eta h\omega_0}{2\gamma T_0^2 \omega_s} \underbrace{\left[\mathrm{Re} Z_{\parallel}(h\omega_0 + \omega_s) - \mathrm{Re} Z_{\parallel}(h\omega_0 - \omega_s)\right]}_{\Delta\left[\mathrm{Re} Z_{\parallel}(h\omega_0)\right]}$$



Stability criterion → $\eta \cdot \Delta\left[\mathrm{Re} Z_{\parallel}(h\omega_0)\right] < 0$

The Robinson instability

Stability criterion → $\eta \cdot \Delta\left[\mathrm{Re}Z_{||}(h\omega_0)\right] < 0$

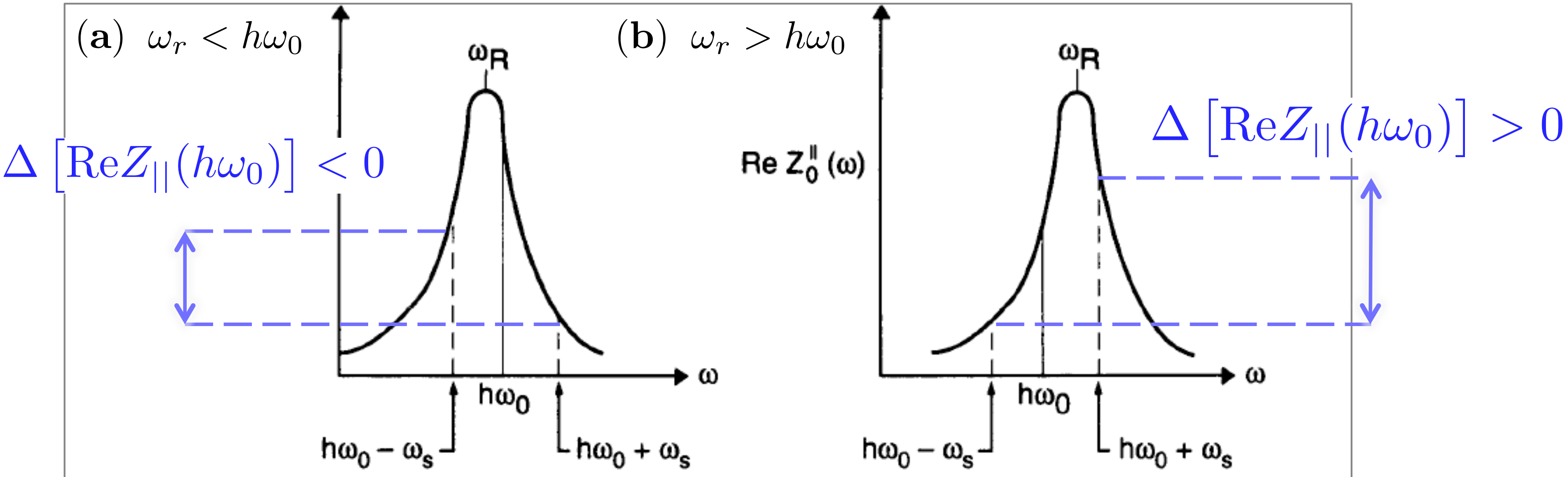


Figure 4.4. Illustration of the Robinson stability criterion. The rf fundamental mode is detuned so that ω_R is (a) slightly below $h\omega_0$ and (b) slightly above $h\omega_0$. (a) is Robinson damped above transition and antidamped below transition. (b) is antidamped above transition and damped below transition.

	$\omega_r < h\omega_0$	$\omega_r > h\omega_0$
Above transition (η>0)	stable	unstable
Below transition (η<0)	unstable	stable

The Robinson instability

$$\tau^{-1} = \mathrm{Im}(\Omega - \omega_s) = \left(\frac{e^2}{m_0 c^2}\right) \frac{N\eta}{2\gamma T_0^2 \omega_s} \sum_{p=-\infty}^{\infty} (p\omega_0 + \omega_s) \mathrm{Re} Z_{\parallel}(p\omega_0 + \omega_s)$$

- ⇒ Other types of impedances can also cause instabilities through the Robinson mechanism
- ⇒ However, a smooth broad-band impedance with no narrow structures on the ω_0 scale cannot give rise to an instability
  - ✓ Physically, this is clear, because the absence of structure on ω_0 scale in the spectrum implies that the wake has fully decayed in one turn time and the driving term in the equation of motion also vanishes

$$\sum_{p=-\infty}^{\infty} (p\omega_0 + \omega_s) \mathrm{Re} Z_{\parallel}(p\omega_0 + \omega_s) \rightarrow \frac{1}{\omega_0} \int_{-\infty}^{\infty} \omega \mathrm{Re} Z_{\parallel}(\omega) d\omega \rightarrow 0$$

One-particle models

- ❑ Longitudinal plane → Robinson instability
- ❑ Transverse plane → Rigid dipole instability

The rigid bunch instability

– To illustrate the rigid bunch instability we will use some simplifications:
  - ⇒ The bunch is point-like and feels an external linear force (i.e. it would execute linear betatron oscillations in absence of the wake forces)
  - ⇒ Longitudinal motion is neglected
  - ⇒ Smooth approximation → constant focusing + distributed wake

– In a similar fashion as was done for the Robinson instability in the longitudinal plane we want to
  - ⇒ Calculate the betatron tune shift due to the wake
  - ⇒ Derive possible conditions for the excitation of an unstable motion

The rigid bunch instability

– To illustrate the rigid bunch instability we will use some simplifications:
  - ⇒ The bunch is point-like and feels an external linear force (i.e. it would execute linear betatron oscillations in absence of the wake forces)
  - ⇒ Longitudinal motion is neglected
  - ⇒ Smooth approximation → constant focusing + distributed wake

$$\frac{d^2y}{ds^2} + \left(\frac{\omega_\beta}{c}\right)^2 y = -\left(\frac{e^2}{m_0c^2}\right)\frac{N}{\gamma C}\sum_{k=-\infty}^{\infty} y(s-kC)W_y(kC)$$

$$y \propto \exp\left(\frac{-i\Omega s}{c}\right) \Rightarrow \Omega^2 - \omega_\beta^2 = \frac{Ne^2}{m_0\gamma C}\sum_{k=-\infty}^{\infty}\exp\left(ik\Omega T_0\right)W_y(kC)$$

$$= -i\frac{Ne^2}{m_0\gamma C T_0}\sum_{p=-\infty}^{\infty} Z_y(p\omega_0+\Omega)$$

Comes from the definition of Z_y

The rigid bunch instability

⇒ We assume a small deviation from the betatron tune
⇒ Re(Ω - ω_β) → Betatron tune shift
⇒ Im(Ω - ω_β) → Growth/damping rate, if it is positive there is an instability!

$$\Omega^2 - \omega_\beta^2 \approx 2\omega_\beta \cdot (\Omega - \omega_\beta)$$

$$\frac{1}{4\pi}\left[\beta_y \frac{eI_b \mathrm{Im}(Z_y^{\mathrm{eff}})}{E}\right] = \frac{1}{4\pi}\oint \beta_y(s)\Delta k(s) ds$$

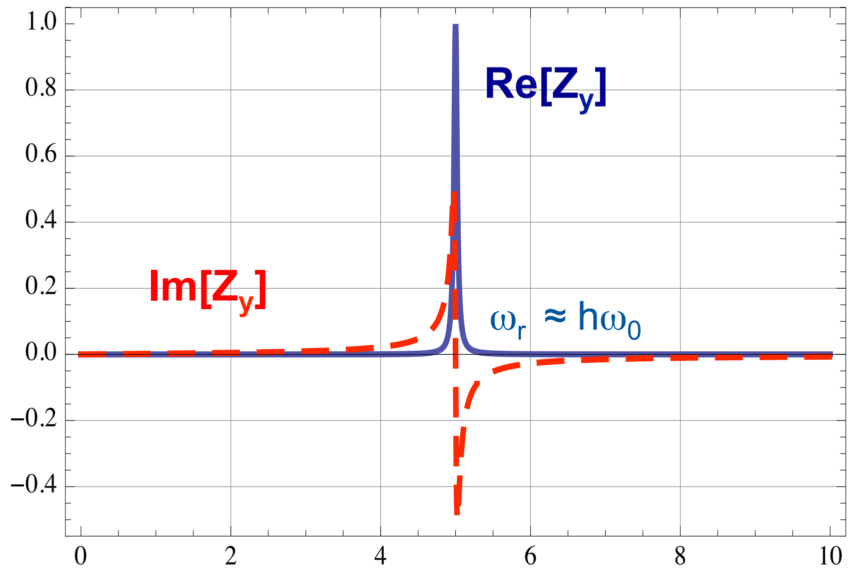
$$\frac{\mathrm{Re}\,(\Omega - \omega_\beta)}{\omega_0} = \Delta\nu_y \approx \boxed{\frac{Ne^2\beta_y}{4\pi m_0 \gamma c C}\sum_{p=-\infty}^{\infty} \mathrm{Im}\left[Z_y(p\omega_0 + \omega_\beta)\right]}$$

$$\mathrm{Im}\,(\Omega - \omega_\beta) = \tau_y^{-1} \approx -\frac{Ne^2\beta_y}{2m_0\gamma C^2}\sum_{p=-\infty}^{\infty} \mathrm{Re}\left[Z_y(p\omega_0 + \omega_\beta)\right]$$

The rigid bunch instability

$$\mathrm{Im}\left(\Omega-\omega_\beta\right)=\tau_y^{-1}\approx-\frac{Ne^2\beta_y}{2m_0\gamma C^2}\sum_{p=-\infty}^{\infty}\mathrm{Re}\left[Z_y(p\omega_0+\omega_\beta)\right]$$

⇒ We assume the impedance to be peaked at a frequency ω_r close to $h\omega_0$ (e.g. RF cavity fundamental mode or HOM)



The rigid bunch instability

$$\mathrm{Im}\left(\Omega - \omega_\beta\right) = \tau_y^{-1} \approx -\frac{Ne^2\beta_y}{2m_0\gamma C^2}\sum_{p=-\infty}^{\infty}\mathrm{Re}\left[Z_y(p\omega_0+\omega_\beta)\right]$$

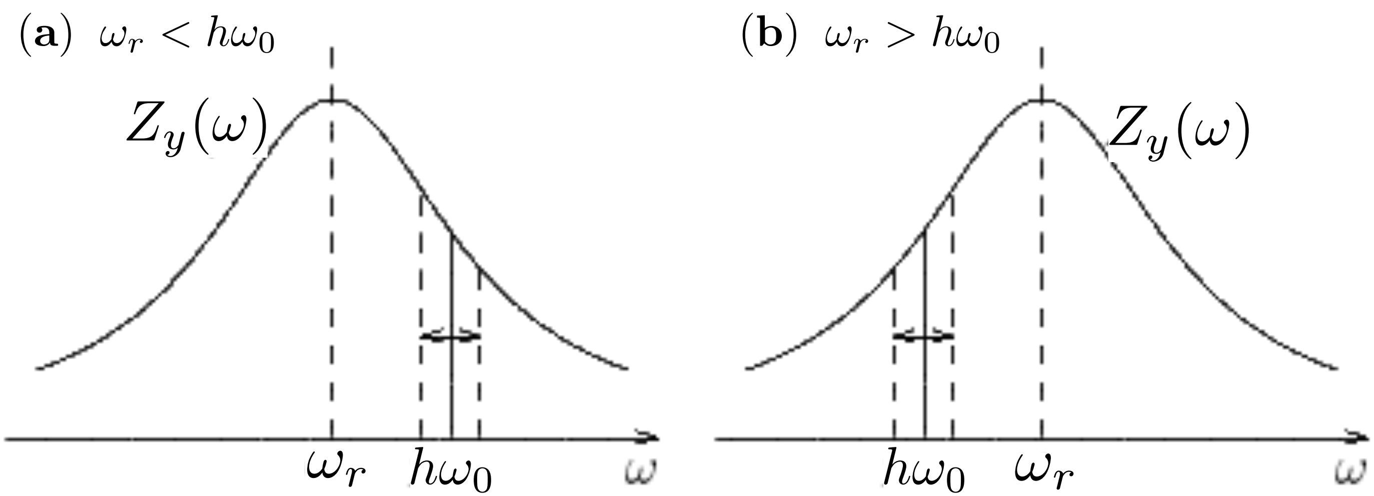
⇒ We assume the impedance to be peaked at a frequency ω_r close to $h\omega_0$ (e.g. RF cavity fundamental mode or HOM)

⇒ Defining the tune $\nu_y = n_y + \Delta_{\beta y}$ with $-0.5 < \Delta_{\beta y} < 0.5$, we can easily express the only two leading terms left in the summation at the RHS of the equation for the growth rate

$$\tau_y^{-1} \approx -\frac{Ne^2\beta_y}{2m_0\gamma C^2}\left(\mathrm{Re}\left[Z_y(h\omega_0+\Delta_{\beta y}\omega_0)\right] - \mathrm{Re}\left[Z_y(h\omega_0-\Delta_{\beta y}\omega_0)\right]\right)$$

The rigid bunch instability

$$\tau_y^{-1} \approx -\frac{Ne^2\beta_y}{2m_0\gamma C^2}\left(\mathrm{Re}\left[Z_y(h\omega_0+\Delta_{\beta y}\omega_0)\right]-\mathrm{Re}\left[Z_y(h\omega_0-\Delta_{\beta y}\omega_0)\right]\right)$$



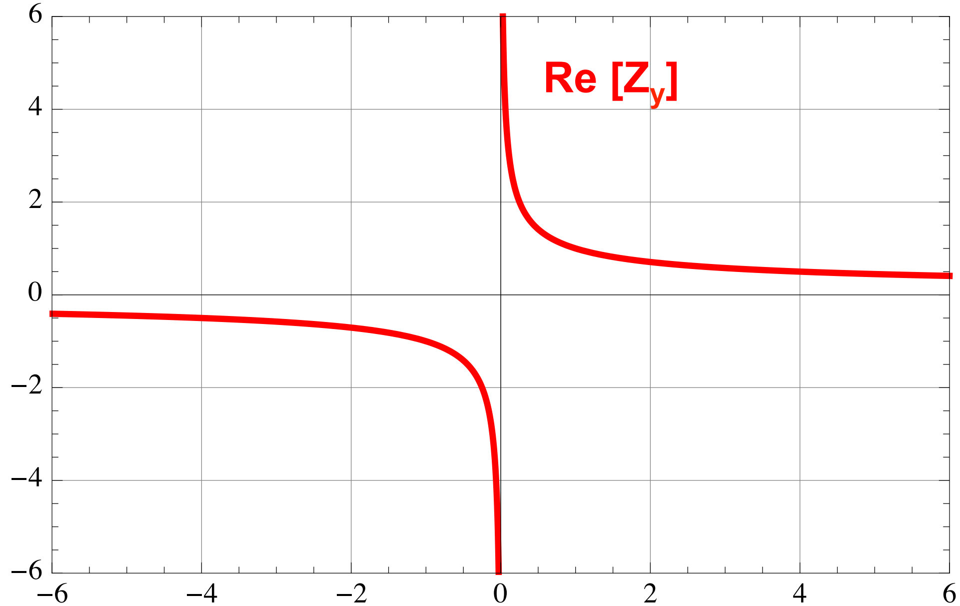
	$\omega_r < h\omega_0$	$\omega_r > h\omega_0$
Tune above integer ($\Delta_{by}>0$)	unstable	stable
Tune below integer ($\Delta_{by}<0$)	stable	unstable

The rigid bunch instability

$$\mathrm{Im}\left(\Omega-\omega_\beta\right)=\tau_y^{-1}\approx-\frac{Ne^2\beta_y}{2m_0\gamma C^2}\sum_{p=-\infty}^{\infty}\mathrm{Re}\left[Z_y(p\omega_0+\omega_\beta)\right]$$

⇒ We assume the impedance to be of resistive wall type, i.e. strongly peaked in the very low frequency range (→ 0)

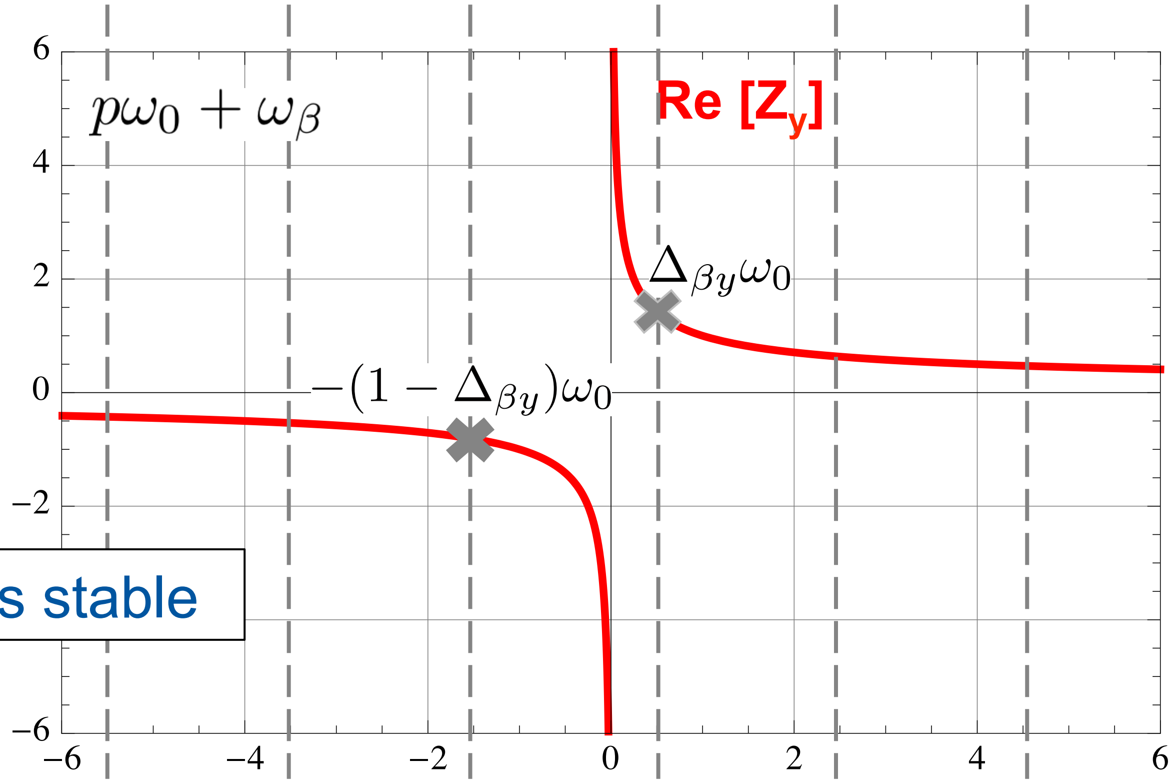
⇒ Using the same definitions for the tune as before, we can easily express the only two leading terms left in the summation at the RHS of the equation for the growth rate



The rigid bunch instability

⇒ Using the same definitions for the tune as before, we can easily express the only two leading terms left in the summation at the RHS of the equation for the growth rate

$$0 < \Delta_{\beta y} < 0.5$$



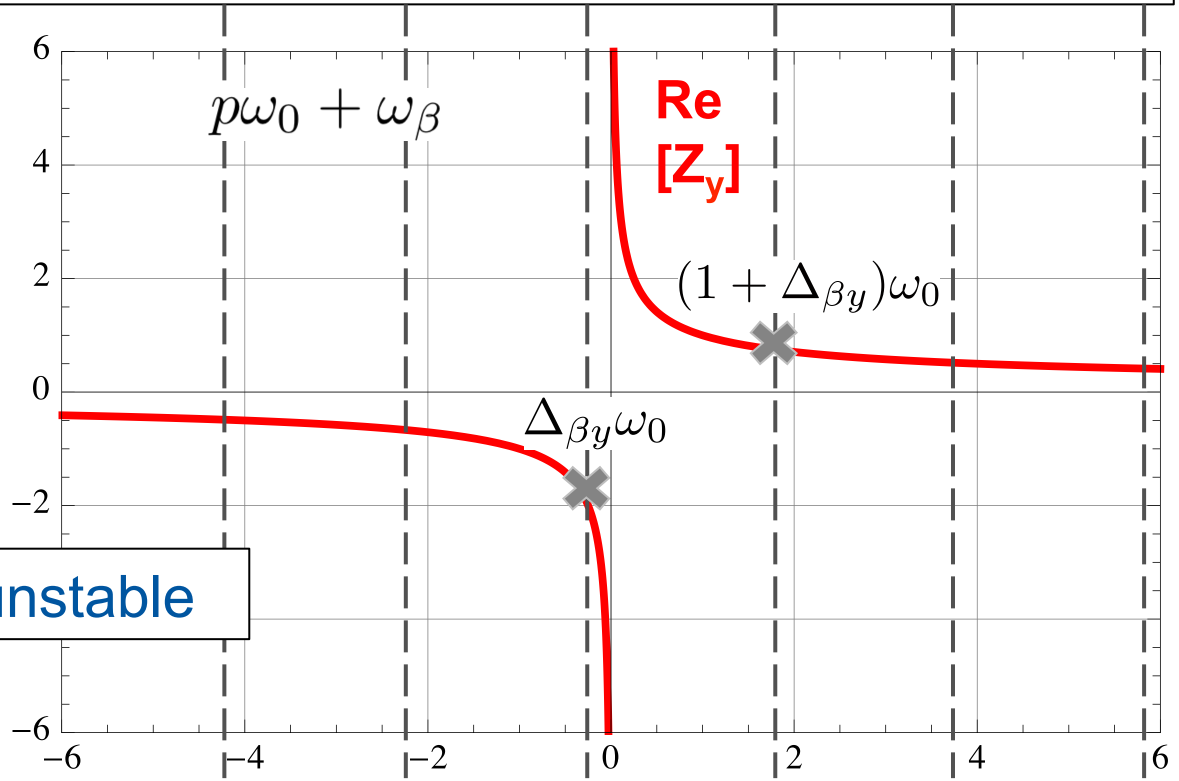
Always stable

$$\tau_y^{-1} \approx -\frac{Ne^2\beta_y}{2m_0\gamma C^2}\left(\mathrm{Re}\left[Z_y(\Delta_{\beta y}\omega_0)\right] - \mathrm{Re}\left[Z_y((1-\Delta_{\beta y})\omega_0)\right]\right) < 0$$

The rigid bunch instability

⇒ Using the same definitions for the tune as before, we can easily express the only two leading terms left in the summation at the RHS of the equation for the growth rate

$$-0.5 < \Delta_{\beta y} < 0$$



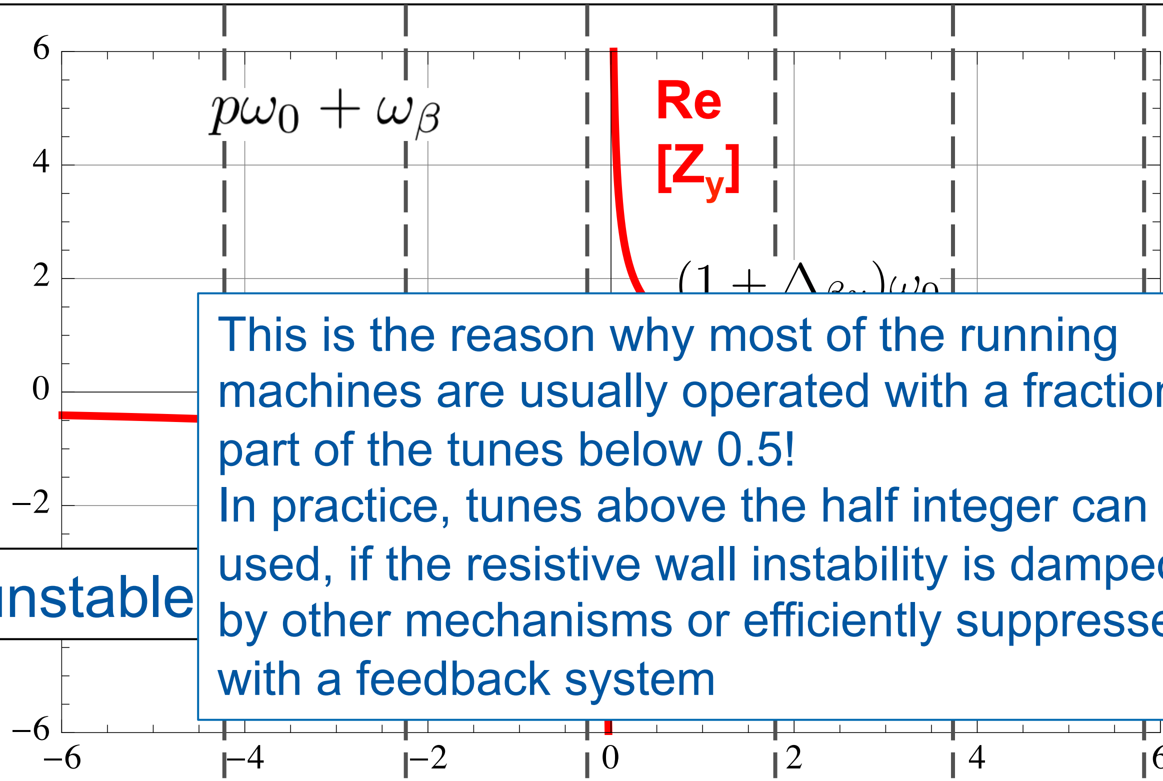
Always unstable

$$\tau_y^{-1} \approx -\frac{Ne^2\beta_y}{2m_0\gamma C^2}\left(\mathrm{Re}\left[Z_y((1+\Delta_{\beta y})\omega_0)\right] - \mathrm{Re}\left[Z_y(-\Delta_{\beta y}\omega_0)\right]\right) > 0$$

The rigid bunch instability

⇒ Using the same definitions for the tune as before, we can easily express the only two leading terms left in the summation at the RHS of the equation for the growth rate

$$-0.5 < \Delta_{\beta y} < 0$$



Always unstable

This is the reason why most of the running machines are usually operated with a fractional part of the tunes below 0.5!
In practice, tunes above the half integer can be used, if the resistive wall instability is damped by other mechanisms or efficiently suppressed with a feedback system

$$\tau_y^{-1} \approx -\frac{Ne^2\beta_y}{2m_0\gamma C^2}\left(\mathrm{Re}\left[Z_y((1+\Delta_{\beta y})\omega_0)\right] - \mathrm{Re}\left[Z_y(-\Delta_{\beta y}\omega_0)\right]\right) > 0$$

Two-particle models

- ❑ Transverse plane → Strong head-tail instability
- ❑ Transverse plane → Head-tail instability

Two-particle models

- Transverse plane → Strong head-tail instability
- Transverse plane → Head-tail instability

The strong head-tail instability

- To illustrate TMCI we will need to make use of some simplifications:
  - ⇒ The bunch is represented through two particles carrying half the total bunch charge and placed in opposite phase in the longitudinal phase space
  - ⇒ They both feel external linear focusing in all three directions (i.e. linear betatron focusing + linear synchrotron focusing).
  - ⇒ Zero chromaticity ($Q'_{x,y}=0$)
  - ⇒ Constant transverse wake left behind by the leading particle
  - ⇒ Smooth approximation → constant focusing + distributed wake

- We will
  - ⇒ Calculate a stability condition (threshold) for the transverse motion
  - ⇒ Have a look at the excited oscillation modes of the centroid

The strong head-tail instability

- To illustrate TMCI we will need to make use of some simplifications:
  - ⇒ The bunch is represented through two particles carrying half the total bunch charge and placed in opposite phase in the longitudinal phase space
  - ⇒ They both feel external linear focusing in all three directions (i.e. linear betatron focusing + linear synchrotron focusing)
  - ⇒ Zero chromaticity ($Q'_{x,y}=0$)
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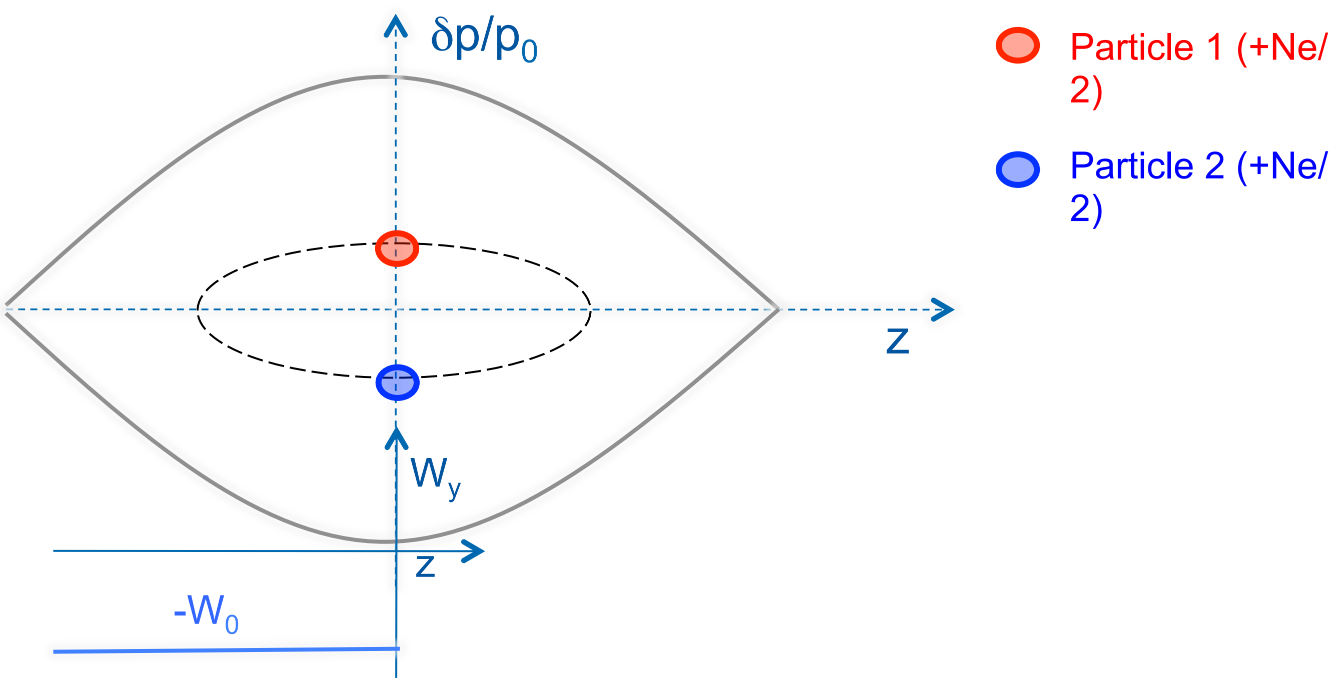

The strong head-tail instability

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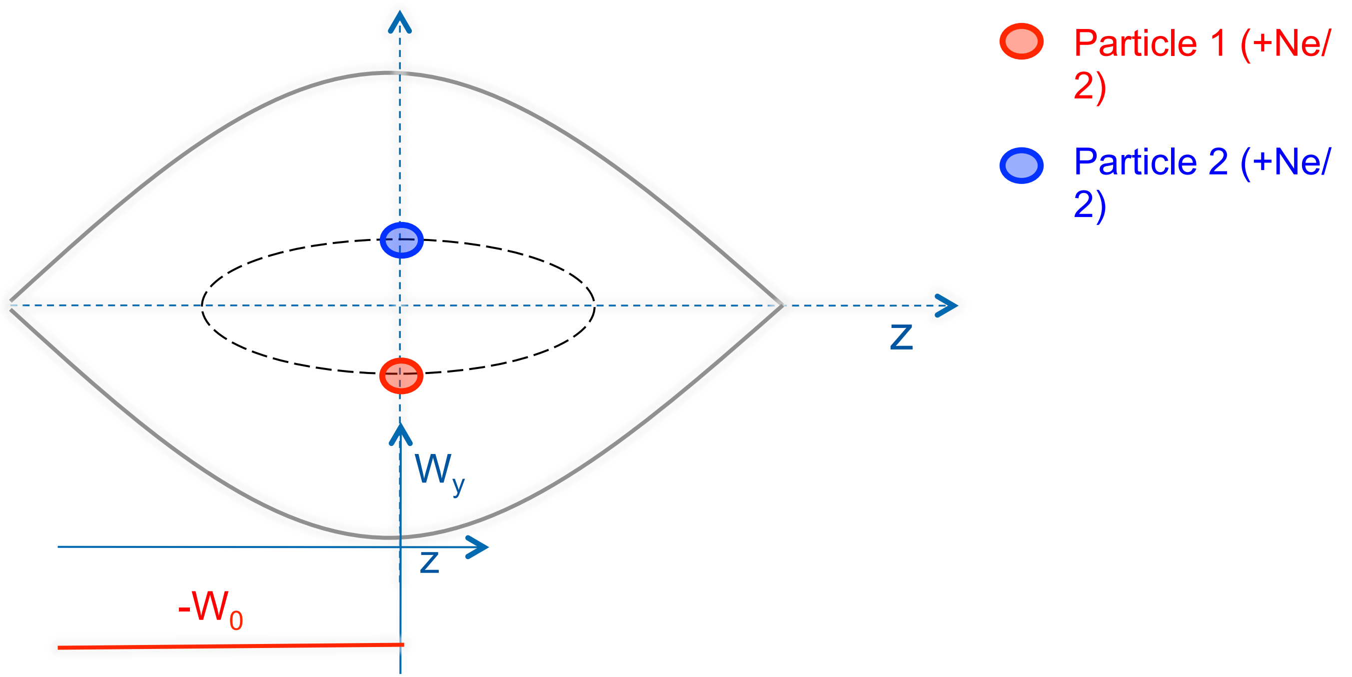
  - ⇒ Constant transverse wake left behind by the leading particle
  - ⇒ Smooth approximation → constant focusing + distributed wake



The strong head-tail instability

⇒ During the first half of the synchrotron motion, particle 1 is leading and executes free betatron oscillations, while particle 2 is trailing and feels the defocusing wake of particle 1

$$\begin{cases} \dfrac{d^2 y_1}{ds^2} + \left(\dfrac{\omega_\beta}{c}\right)^2 y_1 = 0 \\ \dfrac{d^2 y_2}{ds^2} + \left(\dfrac{\omega_\beta}{c}\right)^2 y_2 = \left(\dfrac{e^2}{m_0 c^2}\right) \dfrac{N W_0}{2\gamma C} y_1(s) \end{cases} \qquad 0 < s < \frac{\pi c}{\omega_s}$$

The strong head-tail instability

⇒ During the first half of the synchrotron motion, particle 1 is leading and executes free betatron oscillations, while particle 2 is trailing and feels the defocusing wake of particle 1

⇒ During the second half of the synchrotron period, the situation is reversed

$$\left\{\begin{array}{l} \dfrac{d^2 y_1}{ds^2} + \left(\dfrac{\omega_\beta}{c}\right)^2 y_1 = \left(\dfrac{e^2}{m_0 c^2}\right) \dfrac{N W_0}{2\gamma C} y_2(s) \\[2ex] \dfrac{d^2 y_2}{ds^2} + \left(\dfrac{\omega_\beta}{c}\right)^2 y_2 = 0 \end{array}\right. \qquad \frac{\pi c}{\omega_s} < s < \frac{2\pi c}{\omega_s}$$

The strong head-tail instability

- ⇒ We solve with respect to the complex variables defined below during the first half of synchrotron period
- ⇒ $y_1(s)$ is a free betatron oscillation
- ⇒ $y_2(s)$ is the sum of a free betatron oscillation plus a driven oscillation with $y_1(s)$ being its driving term

$$\tilde{y}_{1,2}(s) = y_{1,2}(s) + i\frac{c}{\omega_\beta} y'_{1,2}(s)$$

$$\tilde{y}_1(s) = \tilde{y}_1(0)\exp\left(\frac{-i\omega_\beta s}{c}\right)$$

$$\tilde{y}_2(s) = \underbrace{\tilde{y}_2(0)\exp\left(-\frac{i\omega_\beta s}{c}\right)}_{\text{Free oscillation term}} + \underbrace{i\frac{Ne^2 W_0}{4m_0\gamma c C\omega_\beta}\left[\frac{c}{\omega_\beta}\tilde{y}_1^*(0)\sin\left(\frac{\omega_\beta s}{c}\right) + \tilde{y}_1(0) s\exp\left(-\frac{i\omega_\beta s}{c}\right)\right]}_{\text{Driven oscillation term}}$$

The strong head-tail instability

$$\tilde{y}_1\left(\frac{\pi c}{\omega_s}\right) = \tilde{y}_1(0)\exp\left(-\frac{i\pi\omega_\beta}{\omega_s}\right)$$

$$\tilde{y}_2\left(\frac{\pi c}{\omega_s}\right) = \tilde{y}_2(0)\exp\left(-\frac{i\pi\omega_\beta}{\omega_s}\right) +$$

$$+i\frac{Ne^2W_0}{4m_0\gamma cC\omega_\beta}\left[\cancel{\frac{c}{\omega_\beta}\tilde{y}_1^*(0)\sin\left(\frac{\pi\omega_\beta}{\omega_s}\right)} + \tilde{y}_1(0)\left(\frac{\pi c}{\omega_s}\right)\exp\left(-\frac{i\pi\omega_\beta}{\omega_s}\right)\right]$$

⇒ Second term in RHS equation for $y_2(s)$ negligible if $\omega_s \ll \omega_\beta$
⇒ We can now transform these equations into linear mapping across half synchrotron period

$$\begin{pmatrix}\tilde{y}_1\\ \tilde{y}_2\end{pmatrix}_{s=\pi c/\omega_s} = \exp\left(-\frac{i\pi\omega_\beta}{\omega_s}\right)\cdot\begin{pmatrix}1 & 0\\ i\Upsilon & 1\end{pmatrix}\cdot\begin{pmatrix}\tilde{y}_1\\ \tilde{y}_2\end{pmatrix}_{s=0}$$

$$\Upsilon = \frac{\pi Ne^2W_0}{4m_0\gamma C\omega_\beta\omega_s}$$

The strong head-tail instability

⇒ In the second half of synchrotron period, particles 1 and 2 exchange their roles
⇒ We can therefore find the transfer matrix over the full synchrotron period for both particles
⇒ We can analyze the eigenvalues of the two particle system

$$\Upsilon = \frac{\pi N e^2 W_0}{4 m_0 \gamma C \omega_\beta \omega_s}$$

$$\begin{pmatrix} \tilde{y}_1 \\ \tilde{y}_2 \end{pmatrix}_{s=2\pi c/\omega_s} = \exp\left(-\frac{i2\pi\omega_\beta}{\omega_s}\right) \cdot \begin{pmatrix} 1 & i\Upsilon \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ i\Upsilon & 1 \end{pmatrix} \cdot \begin{pmatrix} \tilde{y}_1 \\ \tilde{y}_2 \end{pmatrix}_{s=0}$$

$$\begin{pmatrix} \tilde{y}_1 \\ \tilde{y}_2 \end{pmatrix}_{s=2\pi c/\omega_s} = \exp\left(-\frac{i2\pi\omega_\beta}{\omega_s}\right) \cdot \begin{pmatrix} 1-\Upsilon^2 & i\Upsilon \\ i\Upsilon & 1 \end{pmatrix} \cdot \begin{pmatrix} \tilde{y}_1 \\ \tilde{y}_2 \end{pmatrix}_{s=0}$$

The strong head-tail instability

⇒ Since the product of the eigenvalues is 1, the only condition for stability is that they both be purely imaginary exponentials

⇒ From the second equation for the eigenvalues, it is clear that this is true only when sin(ϕ/2)<1

⇒ This translates into a condition on the beam/wake parameters

$$\lambda_1 \cdot \lambda_2 = 1 \quad \Rightarrow \quad \lambda_{1,2} = \exp(\pm i\phi)$$

$$\lambda_1 + \lambda_2 = 2 - \Upsilon^2 \quad \Rightarrow \quad \sin\left(\frac{\phi}{2}\right) = \frac{\Upsilon}{2}$$

$$\boxed{\Upsilon = \frac{\pi N e^2 W_0}{4 m_0 \gamma C \omega_\beta \omega_s} \leq 2}$$

The strong head-tail instability

$$N \leq N_{\text{threshold}} = \frac{8}{\pi e^2} \frac{p_0 \omega_s}{\beta_y} \left(\frac{C}{W_0} \right)$$

- ⇒ Proportional to p_0 → bunches with higher energy tend to be more stable
- ⇒ Proportional to ω_s → the quicker is the longitudinal motion within the bunch, the more stable is the bunch
- ⇒ Inversely proportional to β_y → the effect of the impedance is enhanced if the kick is given at a location with large beta function

- ⇒ Inversely proportional to the wake per unit length along the ring, W_0/C → a large integrated wake (impedance) lowers the instability threshold

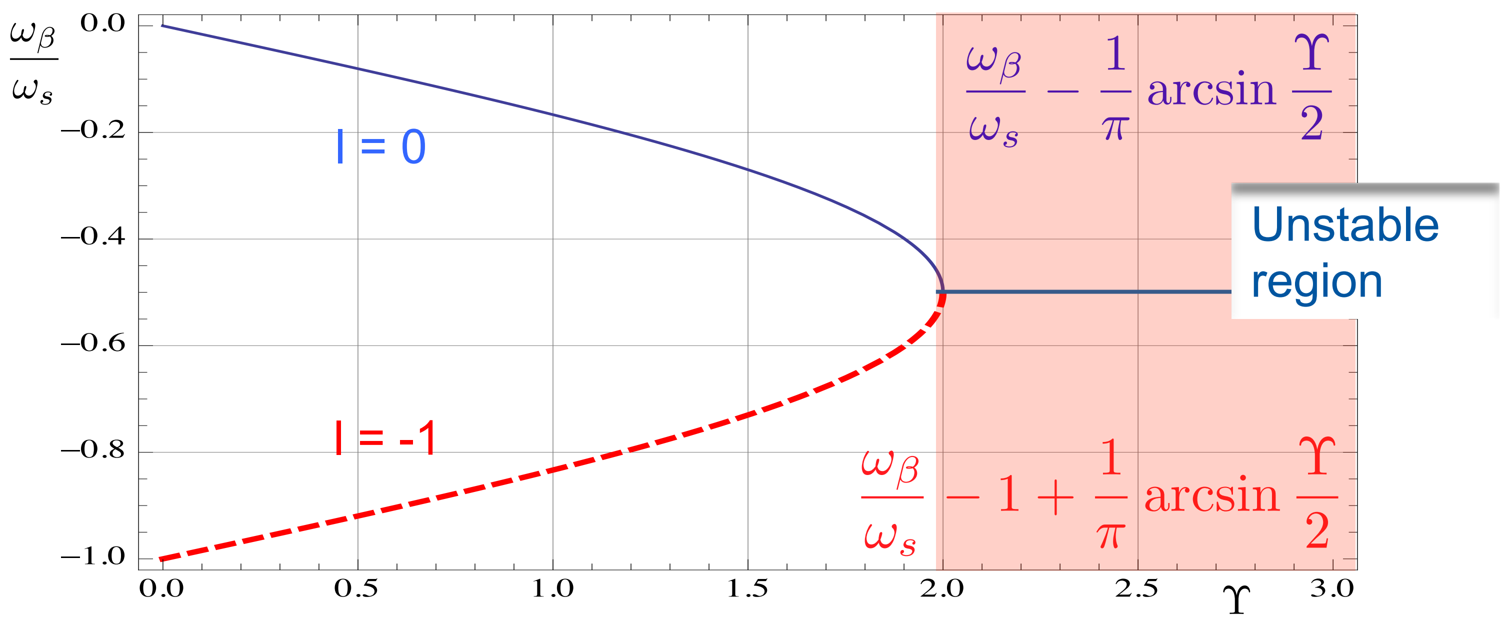
The strong head-tail instability

The evolution of the eigenstates follows:

$$\begin{pmatrix} \tilde{V}_{+n} \\ \tilde{V}_{-n} \end{pmatrix} = \exp\left(-i\frac{2\pi\omega_\beta}{\omega_s}n\right)\cdot\begin{pmatrix} \exp\left[-2i\arcsin\left(\frac{\Upsilon}{2}\right)\cdot n\right] & 0 \\ 0 & \exp\left[2i\arcsin\left(\frac{\Upsilon}{2}\right)\cdot n\right] \end{pmatrix}\begin{pmatrix} \tilde{V}_{+0} \\ \tilde{V}_{-0} \end{pmatrix}$$

Eigenfrequencies: $$\omega_\beta + l\omega_s \pm \frac{\omega_s}{\pi}\arcsin\frac{\Upsilon}{2}$$

They shift with increasing intensity



The strong head-tail instability

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Eigenfrequencies: $$\omega_\beta + l\omega_s \pm \frac{\omega_s}{\pi}\arcsin\frac{\Upsilon}{2}$$

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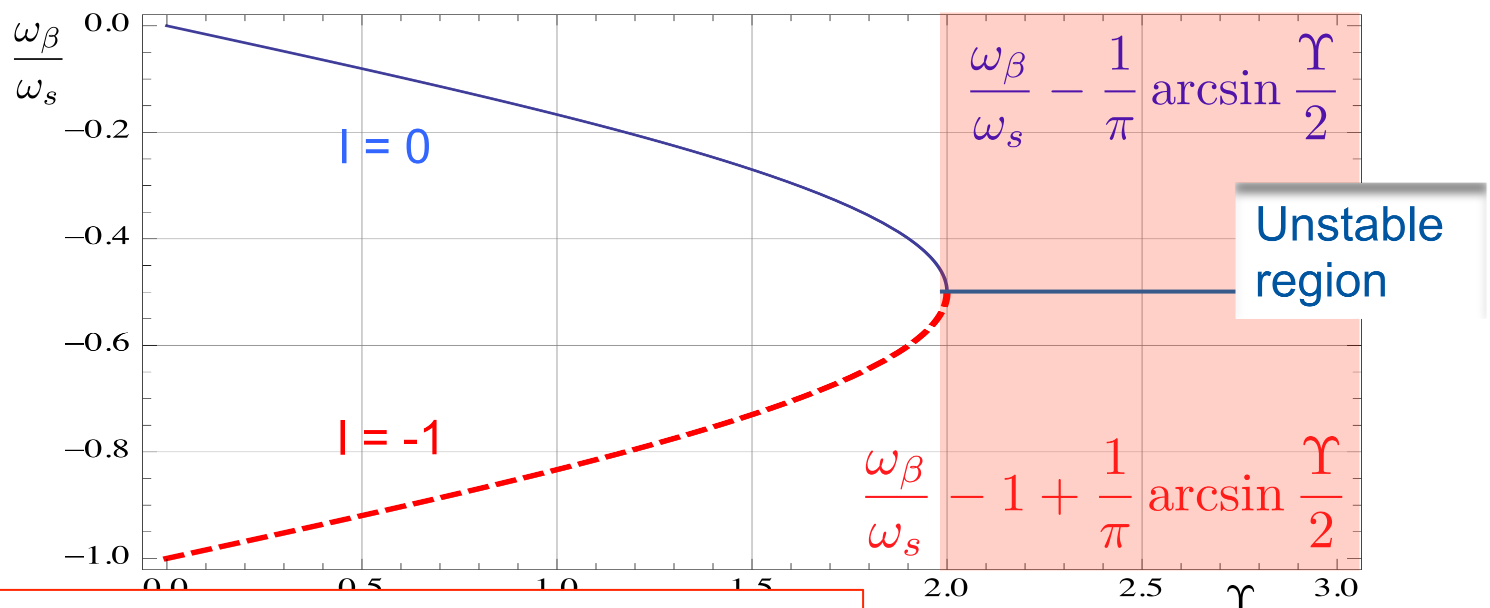
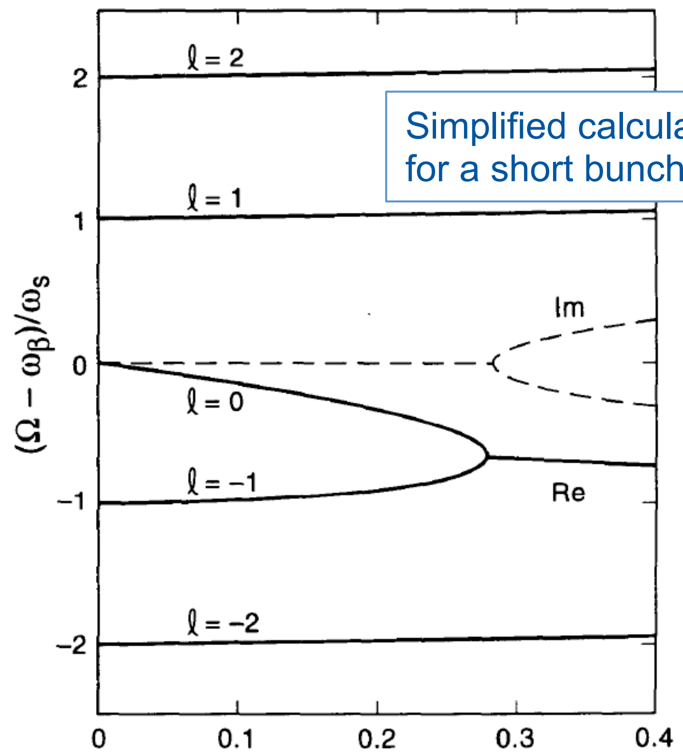


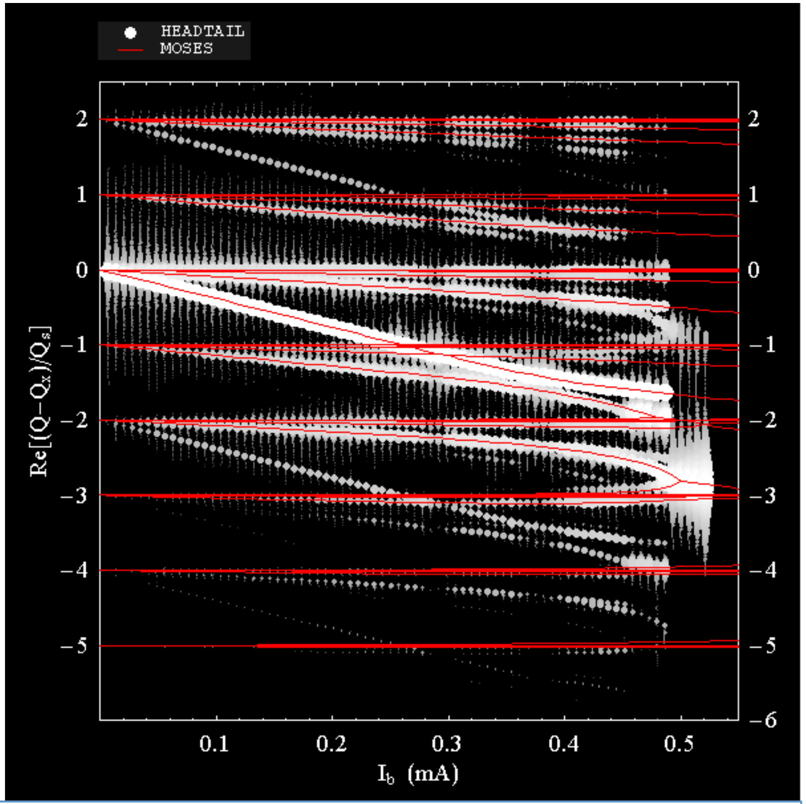
That's the reason why this type of instability is called Transverse Mode Coupling Instability!

The strong head-tail instability

⇒ For a real bunch, modes exhibit a more complicated shift pattern
⇒ The shift of the modes can be calculated via Vlasov equation or can be found through macroparticle simulations

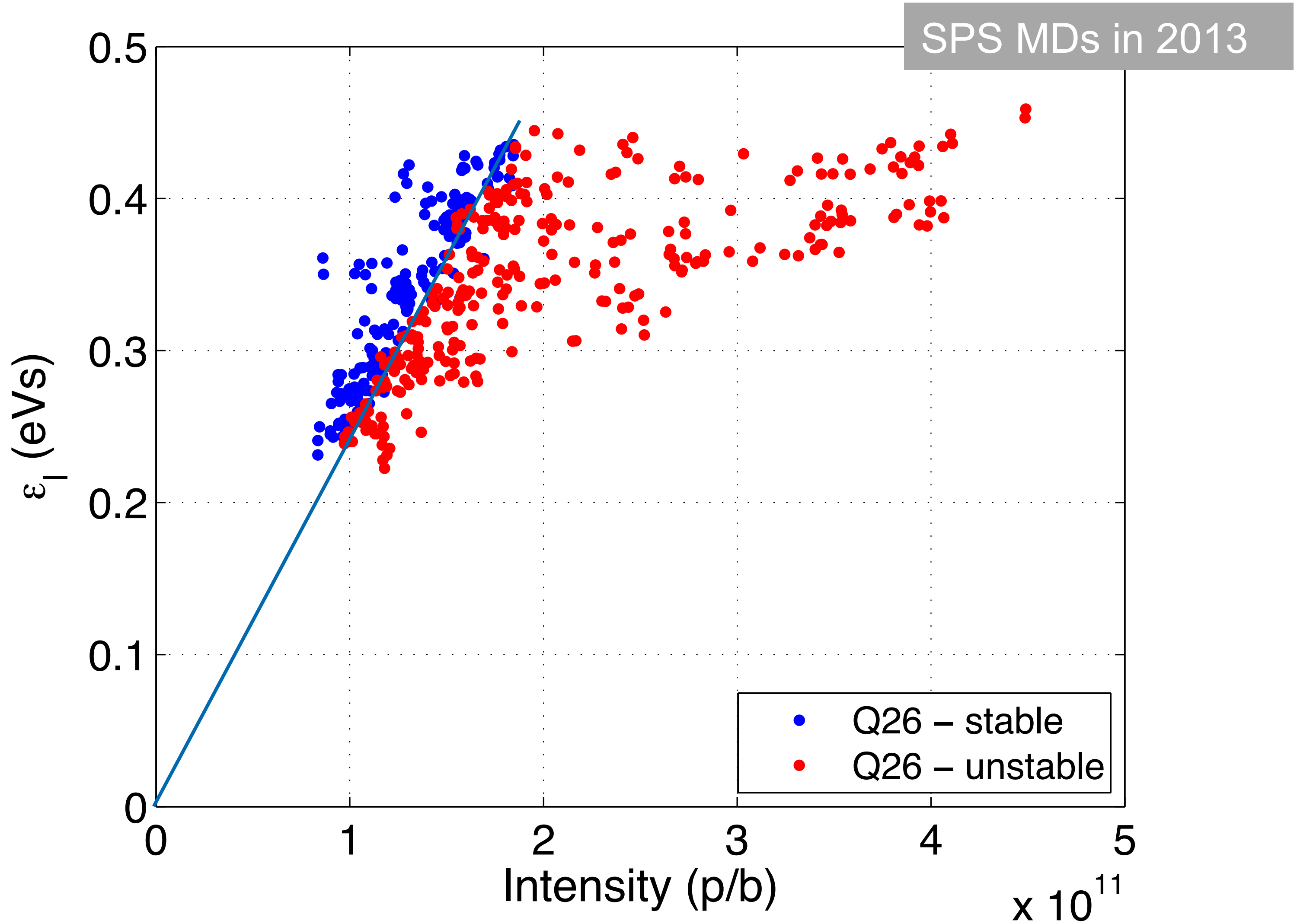


Simplified calculation for a short bunch



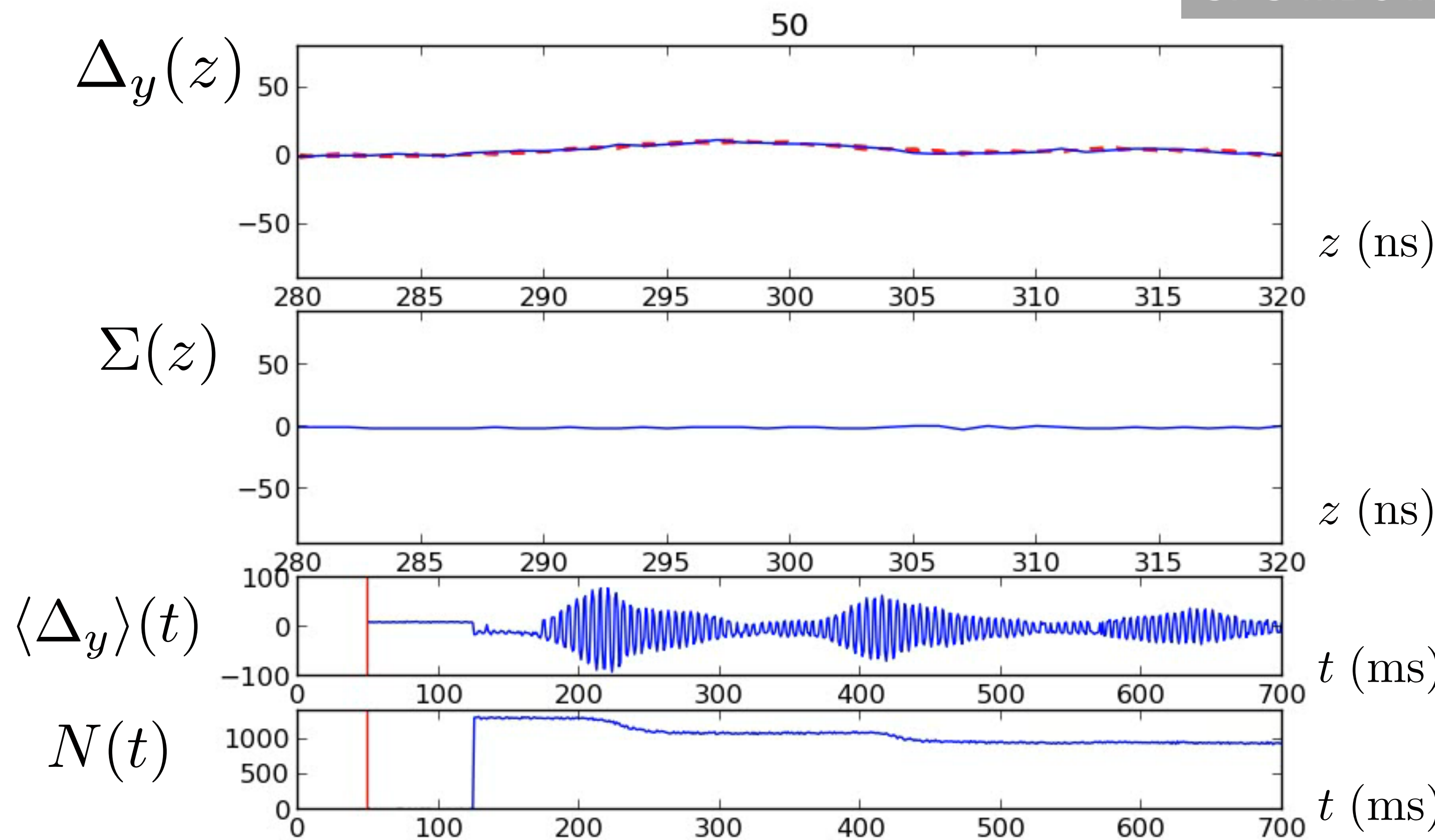
Full calculation for a relatively long SPS bunch (red lines) + macroparticle simulation (white traces)

The strong head-tail instability



The strong head-tail instability

SPS MDs in 2013



Two-particle models

- ❑ Transverse plane → Strong head-tail instability
- ❑ Transverse plane → Head-tail instability

The head-tail instability

⇒ As for the TMCI, during the first half of the synchrotron motion, particle 1 is leading and executes free betatron oscillations, while particle 2 is trailing and feels the defocusing wake of particle 1

⇒ During the second half of the synchrotron period, the situation is reversed

$$
\begin{cases}
\dfrac{d^2 y_1}{ds^2} + \left[\dfrac{\omega_\beta(1+\xi_y\delta(s))}{c}\right]^2 y_1 = 0 \\[2ex]
\dfrac{d^2 y_2}{ds^2} + \left[\dfrac{\omega_\beta(1+\xi_y\delta(s))}{c}\right]^2 y_2 = \left(\dfrac{e^2}{m_0c^2}\right)\dfrac{NW_0}{2\gamma C}y_1(s)
\end{cases}
\qquad 0 < s < \frac{\pi c}{\omega_s}
$$

Difference! → now the frequency of free oscillation is modulated by the momentum spread, δ(s)

The head-tail instability

⇒ Similarly to the solution for the Strong Head Tail Instability, we obtain the transport map

$$\begin{pmatrix} \tilde{y}_1 \\ \tilde{y}_2 \end{pmatrix}_{s=2\pi c/\omega_s} = \begin{pmatrix} i\Upsilon & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} \tilde{y}_1 \\ \tilde{y}_2 \end{pmatrix}_{s=\pi c/\omega_s} = \begin{pmatrix} 1-\Upsilon^2 & i\Upsilon \\ i\Upsilon & 1 \end{pmatrix} \cdot \begin{pmatrix} \tilde{y}_1 \\ \tilde{y}_2 \end{pmatrix}_{s=0}$$

$$\Upsilon = \frac{\pi N e^2 W_0}{4 m_0 \gamma C \omega_\beta \omega_s} \left(1 + i \frac{4 \xi_y \omega_\beta \hat{z}}{\pi c \eta} \right)$$

Complex number!

Weak beam intensity:

$$|\Upsilon| \ll 1$$

$$\lambda_\pm \approx \exp(\pm i \Upsilon)$$

+ mode is “in-phase” mode → the two particles oscillate in phase (ω_β)

- mode is “out-phase” mode → the two particles oscillate in opposition of phase ($\omega_\beta \pm \omega_s$)

The head-tail instability

$$\tau^{-1} = \mathrm{Im}\left(\pm\Upsilon \cdot \frac{\omega_s}{2\pi}\right) = \mp\frac{e^2}{2\pi}\cdot\frac{N\xi_y\hat{z}}{p_0\eta}\left(\frac{W_0}{C}\right)$$

⇒ Inversely proportional to p_0 → bunches with higher energy tend to be less affected by impedances

⇒ Proportional to N → the more intense is the bunch, the more sensitive it is

⇒ Proportional to bunch length → this depends on the chosen shape of the wake

⇒ Proportional to ξ_y → higher chromaticity enhances the head-tail effect

⇒ Inversely proportional to η → faster synchrotron motion stabilizes (lowest rise times close to transition crossing!)

⇒ Proportional to the wake per unit length along the ring, W_0/C → a large integrated wake (impedance) gives a stronger effect

The head-tail instability

$$\tau^{-1} = \mathrm{Im}\left(\pm\Upsilon \cdot \frac{\omega_s}{2\pi}\right) = \mp\frac{e^2}{2\pi} \cdot \frac{N\xi_y \hat{z}}{p_0\eta}\left(\frac{W_0}{C}\right)$$

Mode 0 (+)

	$\xi_y>0$	$\xi_y<0$
Above transition ($\eta>0$)	damped	unstable
Below transition ($\eta<0$)	unstable	damped

Mode 1 (−)

	$\xi_y>0$	$\xi_y<0$
Above transition ($\eta>0$)	unstable	damped
Below transition ($\eta<0$)	damped	unstable

The head-tail instability

- The head-tail instability is unavoidable in the two-particle model
  - Either mode 0 or mode 1 is unstable
  - Growth/damping times are in all cases identical

- Fortunately, the situation is less dramatic in reality
  - The number of modes increases with the number of particles we consider in the model (and becomes infinite in the limit of a continuous bunch)
  - The instability conditions for mode 0 remain unchanged, but all the other modes become unstable with much longer rise times when mode 0 is stable

Mode 0

	$\xi_y>0$	$\xi_y<0$
Above transition ($\eta>0$)	damped	unstable
Below transition ($\eta<0$)	unstable	damped

$$\sum_{l=-\infty}^{\infty} \frac{1}{\tau_l} = 0$$

All modes >0

	$\xi_y>0$	$\xi_y<0$
Above transition ($\eta>0$)	unstable	damped
Below transition ($\eta<0$)	damped	unstable

The head-tail instability

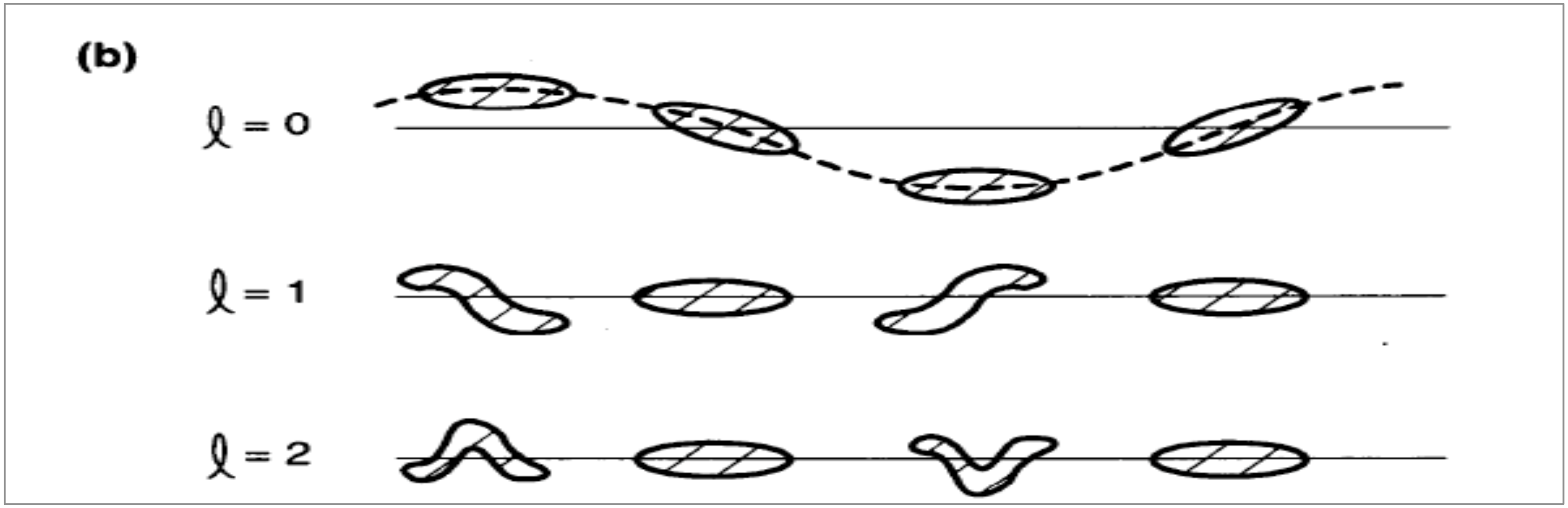
- The head-tail instability is unavoidable in the two-particle model
  - Either mode 0 or mode 1 is unstable
  - Growth/damping times are in all cases identical

- Fortunately, the situation is less dramatic in reality
  - The number of modes increases with the number of particles we consider in the model (and becomes infinite in the limit of a continuous bunch)
  - The instability conditions for mode 0 remain unchanged, but all the other modes become unstable with much longer rise times when mode 0 is stable
  - Therefore, the bunch can be in practice stabilized by using the settings that make mode 0 stable (ξ<0 below transition and ξ>0 above transition) and relying on feedback or Landau damping (refer to W. Herr's lectures) for the other modes

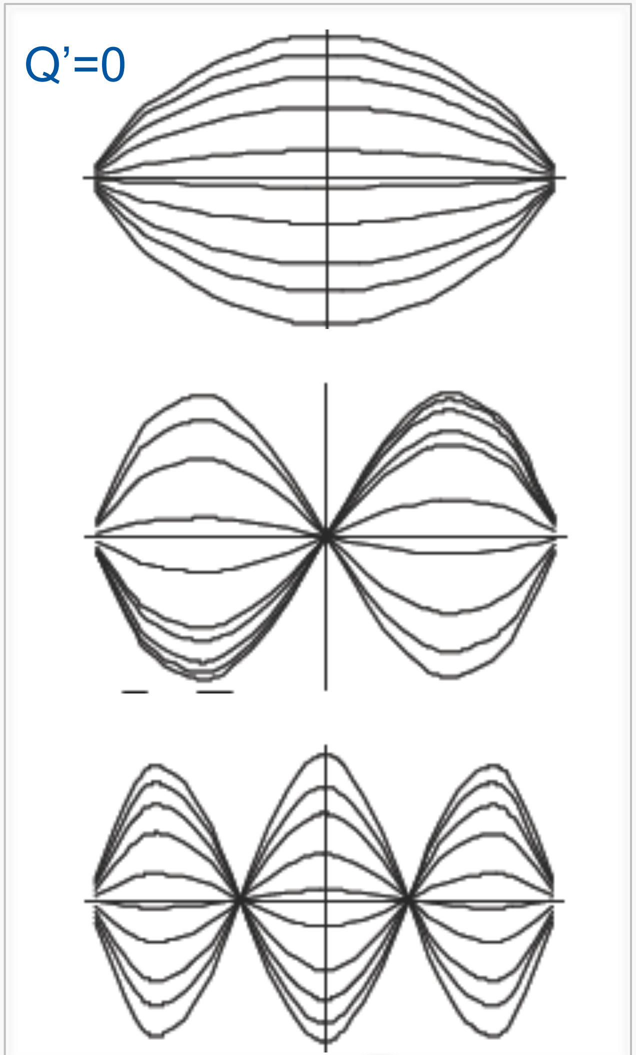
- To be able to study these effects we would need to resort to a more detailed description of the bunch
  - **Vlasov equation (kinetic model)**
  - **Macroparticle simulations**

The head-tail instability
A glance into head-tail mode

- Different transverse head-tail modes correspond to different parts of the bunch oscillating with relative phase differences. E.g.
  - Mode 0 is a rigid bunch mode
  - Mode 1 has head and tail oscillating in counter-phase
  - Mode 2 has head and tail oscillating in phase and the bunch center in opposition



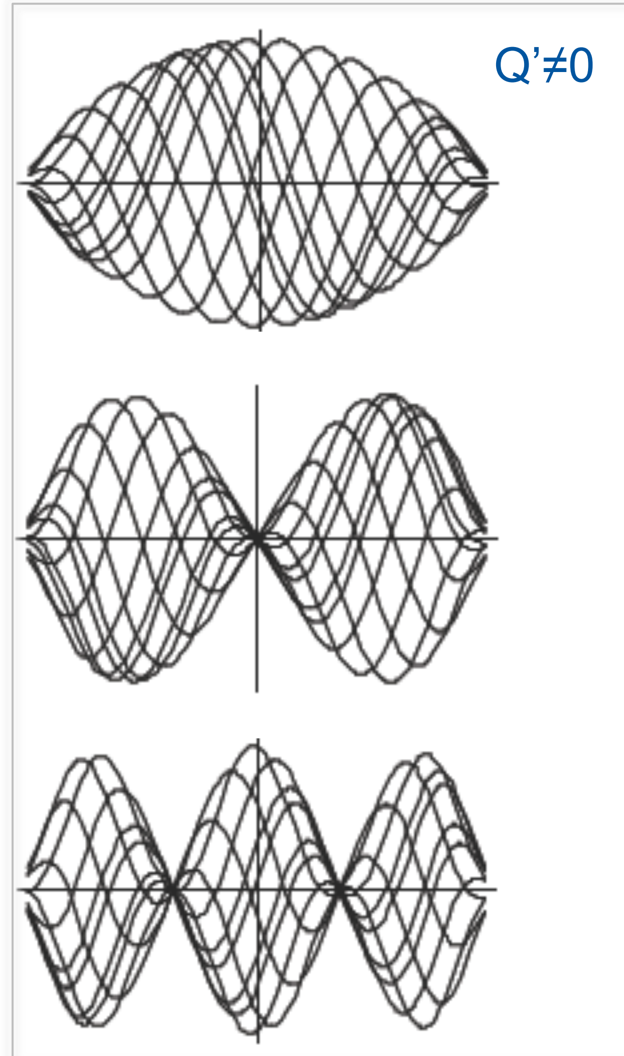
The head-tail instability
How a BPM detects head-tail modes



← *l=0* →

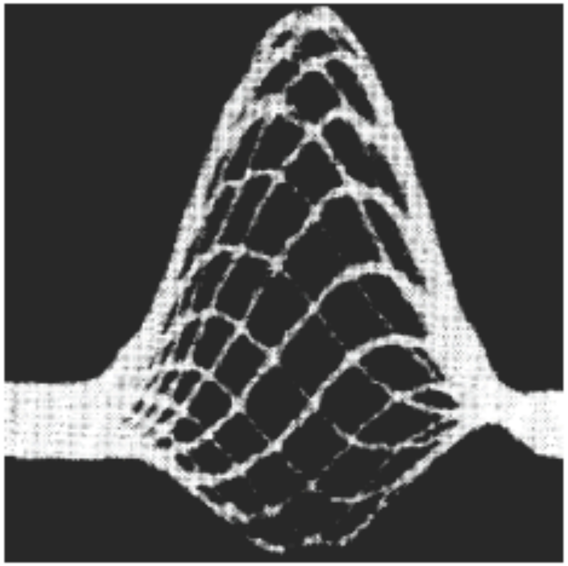
← *l=1* →

← *l=2* →

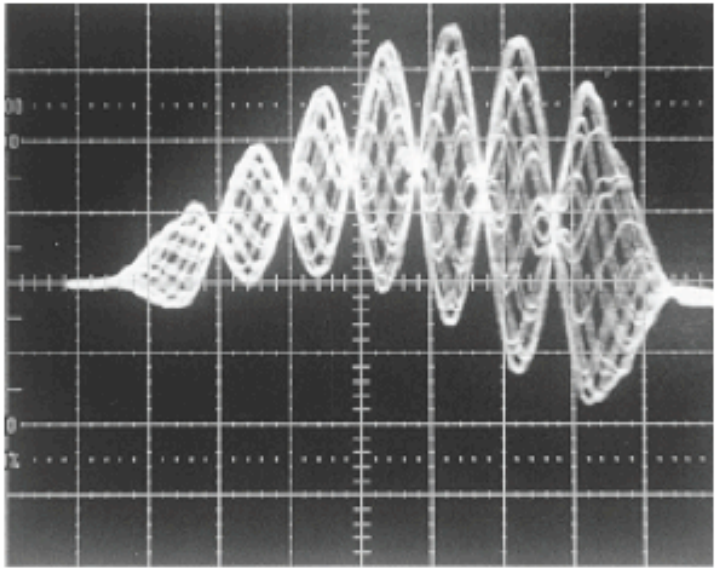


The head-tail instability
Experimental observations (historical)

Observation in the CERN PSB in ~1974
(J. Gareyte and F. Sacherer)



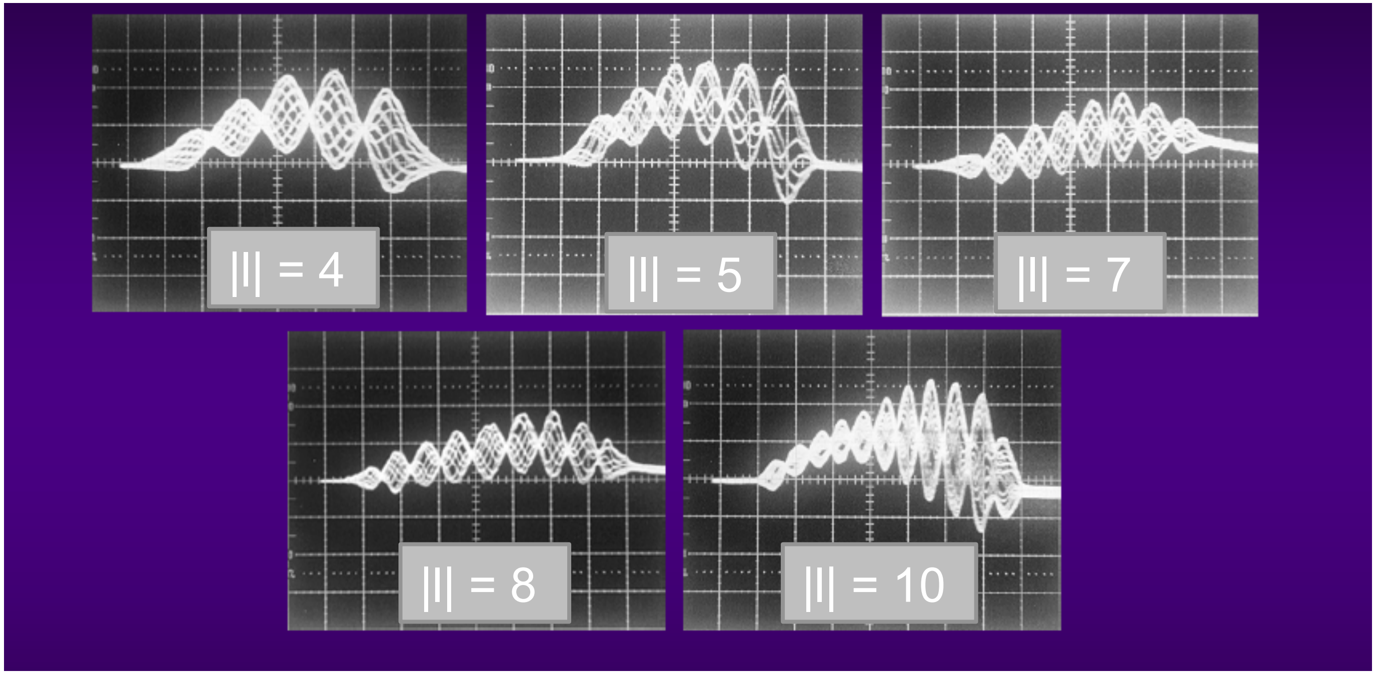
Observation in the CERN PS in 1999



- The mode that gets first excited in the machine depends on
  - The spectrum of the exciting impedance
  - The chromaticity setting
- Head-tail instabilities are a good diagnostics tool to identify and quantify the main impedance sources in a machine

The head-tail instability
Experimental observations

- Higher order head-tail modes ($l \geq 1$) are usually stabilized by tune spread and/or active feedback. However, if a high intensity beam stays in a machine long enough without sufficient tune spread and without feedback, these modes can also slowly grow.
- For example, a high intensity bunch becomes unstable in the CERN-PS over 1.2 s due to resistive wall

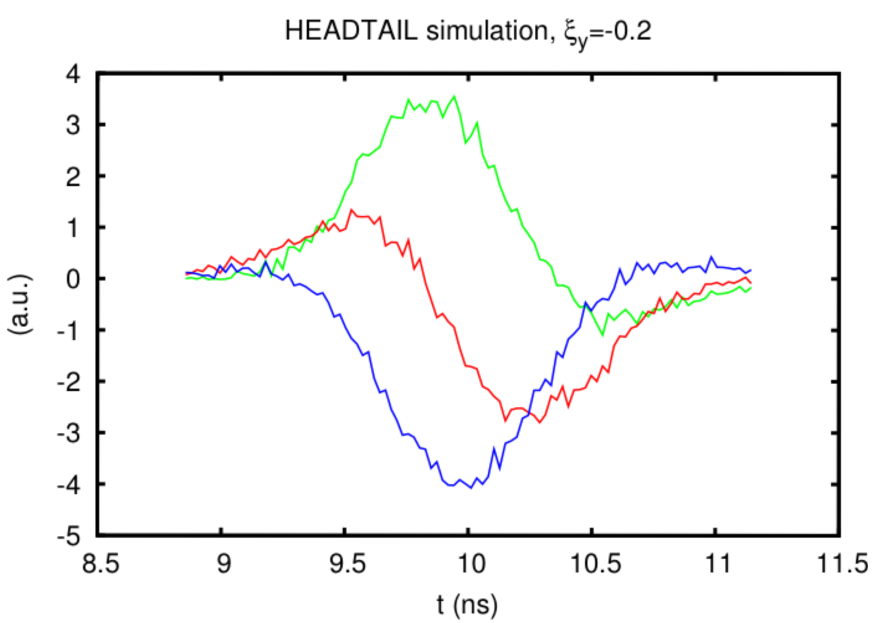


The head-tail instability Experimental observations

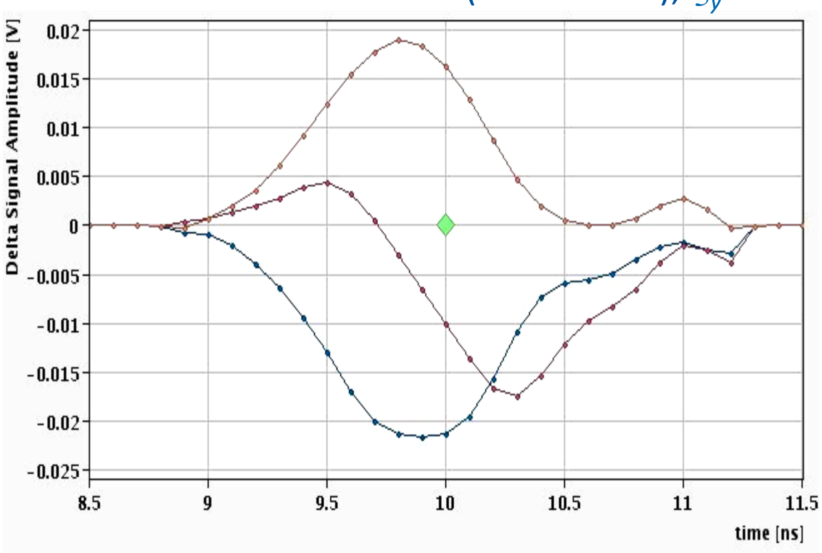
- The fundamental mode of a head-tail instability can be simulated to have a detailed look at the instability evolution for different chromaticity values (assuming the SPS parameters and a simple broad band model for the impedance)

  ⇒ Simulations reproduce what is observed in the machine!

  ⇒ Plots show three consecutive traces of the centroid signal along the bunch while the instability is growing

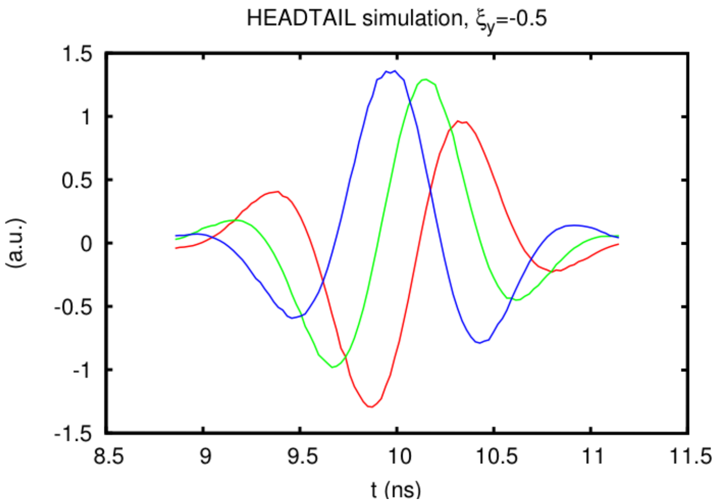


Measurement at the SPS (06.08.2007), ξ_y=-0.2

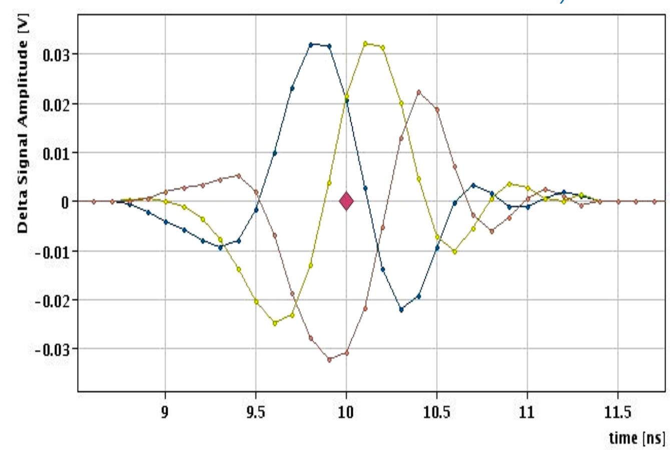


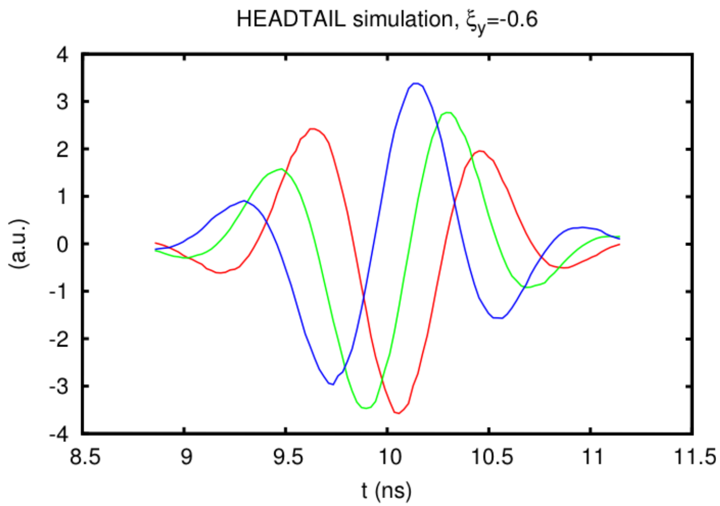
The head-tail instability Experimental observations

- Different values of chromaticity…

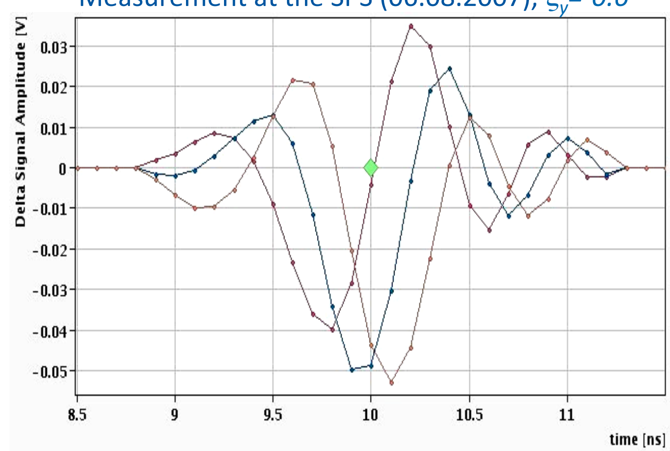


Measurement at the SPS (06.08.2007), ξ_y=-0.5





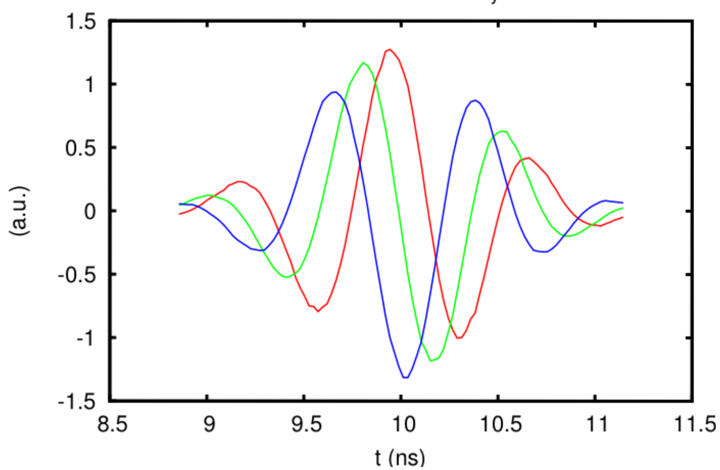
Measurement at the SPS (06.08.2007), ξ_y=-0.6



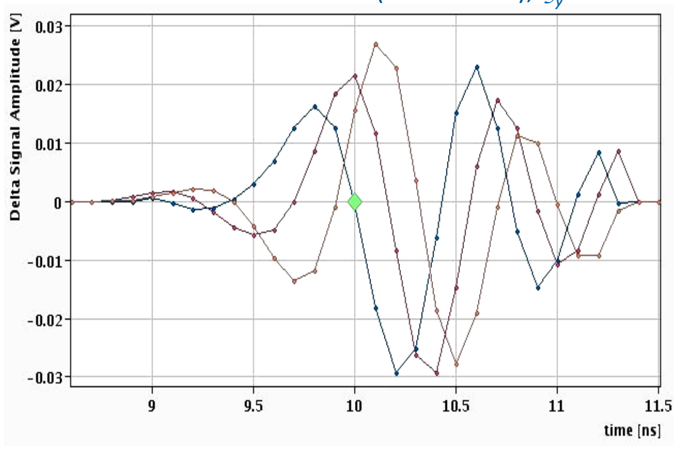
The head-tail instability
Experimental observations

- Different values of chromaticity…

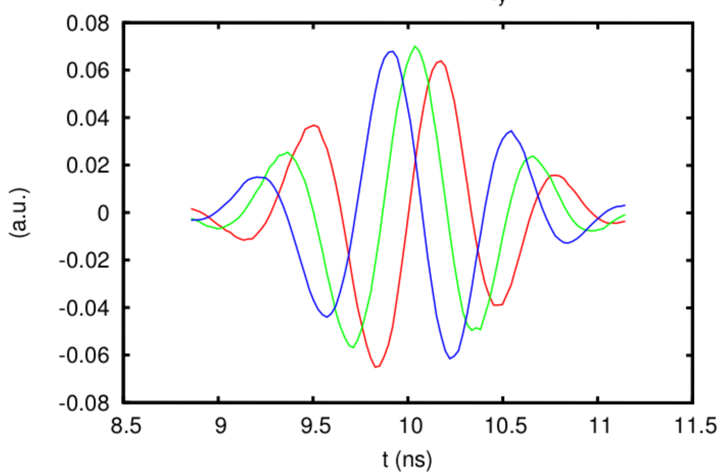
HEADTAIL simulation, ξ_y=-0.7



Measurement at the SPS (06.08.2007), ξ_y=-0.7



HEADTAIL simulation, ξ_y=-0.8



Measurement at the SPS (06.08.2007), ξ_y=-0.8