Perturbation Formalism

Outline

1. Introductory concepts
    - Collective effects
    - Transverse single particle dynamics including systems of many non-interacting particles
    - Longitudinal single particle dynamics including systems of many non-interacting particles
2. Space charge
    - Direct space charge (transverse)
    - Indirect space charge (transverse)
    - Longitudinal space charge

Outline

3. Wake fields and impedance
    - Definition of beam coupling impedance
    - Examples – resonators and resistive wall
    - Energy loss
    - Wake function and wake potential
    - Impedance model of a machine
4. Instabilities – few-particle model
    - Equations of motion
    - Longitudinal plane: Robinson instability
    - Transverse plane: rigid bunch instability, strong head-tail instability, head-tail instability

Outline

5. Instabilities – kinetic theory
   - Introduction to Vlasov equation and perturbation approach
   - Vlasov equation in the longitudinal plane
   - Vlasov equation in the transverse plane
   - Oscillation modes, shift with intensity, instability

Hamiltonian systems

We define a Hamiltonian system (Γ, ω, X) composed of:

- a manifold Γ
- a symplectic form ω
- a Hamiltonian vector field X

where the Hamiltonian vector field X is determined by the condition

$$i_X \omega = dH \quad (\Leftrightarrow \omega(X, Y) = dH(Y))\,.$$

In canonical coordinates:

$$X = J\vec{\nabla}H\,, \quad J = \begin{pmatrix} 0 & \mathbf{1} \\ -\mathbf{1} & 0 \end{pmatrix}.$$

Hamiltonian systems

We can restate the principle of least action as: [a]
the time evolution of the state vectors $(\vec{q}, \vec{p}) \in \Gamma$ is given by the integral curve of X:

$$(\dot{\vec{q}}, \dot{\vec{p}}) = X(\vec{q}, \vec{p}) = J\vec{\nabla} H(\vec{q}, \vec{p})\,.$$

It follows that the time evolution of any function $\psi \in f : \Gamma \to \mathbb{R}$ is given by the Poisson bracket:

$$\dot{\psi} = -[H, \psi] + \partial_t \psi \equiv 0\,.^{b}$$

In particular, any function $\psi(H)$ is a stationary solution (in the narrower sense), as

$$\partial_t \psi(H) = [H, \psi(H)] = 0\,.$$

[a] Fundamental physics, see e.g. Goldstein
[b] The last step follows from Liouville's theorem

Exercise: prove this. Can you also show this just using the algebraic properties of the Poisson bracket?
Hint: In particular, the Poisson bracket acts as a derivation.

Poisson brackets

The Poisson bracket is defined as

$$[F, G] = \frac{\partial F}{\partial q}\frac{\partial G}{\partial p} - \frac{\partial F}{\partial p}\frac{\partial G}{\partial q} \, .$$

It has the following algebraic properties

1. $[F, G] = -[G, F]$ (Anticommutativity)
2. $[F + G, H] = [F, H] + [G, H]$ (Distributivity)
3. $[FG, H] = [F, H]\, G + F\, [G, H]$ (Derivation)
4. $[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0$ (Jacobi identity)

Multi-particle systems

State space Γ (Γ-space) for a multi-particle system of particlenumber N:

- State space vectors:

$$(q_1, q_2, \ldots, q_{3N}, p_1, p_2, \ldots, p_{3N}) \in \Gamma \sim \mathbb{R}^{6N}$$

- Energy function:

$$H \in f : \Gamma \sim \mathbb{R}^{6N} \to \mathbb{R}$$

$$H(q_1, q_2, \ldots, q_{3N}, p_1, p_2, \ldots, p_{3N}) = \sum_{i}^{3N} \left(H_s(q_i, p_i) + \sum_{j \neq i}^{3N} H_c(q_i, q_j) \right)$$

Particlenumber
in the observable universe
$N \approx 10^{80} - 10^{99}$

Particlenumber
in molecular systems
$N \approx 10^{23}$

Particlenumber
in plasmas (beams)
$N \approx 10^{9} - 10^{15}$

Particlenumber
in numerical simulations
$N \approx 10^{6}$

Single particle PDF

Our strategy will be to find a more practical representation for the state space vectors. We start off with the state space vectors and energy function for a multi-particle system of particlenumber N:

- State space vectors:

$$(q_1, q_2, \ldots, q_{3N}, p_1, p_2, \ldots, p_{3N}) \in \Gamma \sim \mathbb{R}^{6N}$$

- Energy function:

$$H(q, p) \in f : \Gamma \sim \mathbb{R}^{6N} \to \mathbb{R}$$

- Time evolution:

$$\partial_t(q_1, q_2, \ldots, q_{3N}, p_1, p_2, \ldots, p_{3N}) = [H, (q_1, q_2, \ldots, q_{3N}, p_1, p_2, \ldots, p_{3N})]$$

Single particle PDF

Our strategy will be to find a more practical representation for the state space vectors. We move from representing a multi-particle state by the state space vectors to the multi-particle probability distribution function...

- State functions:

$$\psi_N(q,p) \in f : \Gamma \sim \mathbb{R}^{6N} \to \mathbb{R}$$

- Energy function:

$$H(q,p) \in f : \Gamma \sim \mathbb{R}^{6N} \to \mathbb{R}$$

- Time evolution:

$$\partial_t \psi_N(q,p) = [H, \psi_N]$$

Single particle PDF

Our strategy will be to find a more practical representation for the state space vectors. We move from representing a multi-particle state by the state space vectors to the multi-particle probability distribution function... from there directly to the single particle probability density function:

- State functions:

$$\psi_1(q,p) \in f : \Gamma \sim \mathbb{R}^6 \to \mathbb{R}$$

- Energy function:

$$H(q,p) \in f : \Gamma \sim \mathbb{R}^6 \to \mathbb{R}$$

- Time evolution:

$$\partial_t \psi_1(q,p) = [H, \psi_1]$$

It turns out that the single particle probability density function is perfectly well suited representation of the original multi-particle state.

Single particle PDF

Doing this rigorous, we would need to start off from the N-particle probability density function

$$\psi_N(q_1, q_2, \ldots, q_{3N}, p_1, p_2, \ldots, p_{3N}, t) \in f : \Gamma \sim \mathbb{R}^{6N} \to \mathbb{R}$$

together with the Liouville equation

$$\partial_t \psi_N = [H, \psi_N]$$

and then move through the full BBGKY-hierarchy (Bogoliubov-Born-Green-Kirkwood-Yvon) to obtain the single-particle probability density function

$$\psi_1(q, p, t) = N \int \psi_N(q, q_2, \ldots, q_{3N}, p, p_2, \ldots, p_{3N})\, dq_2 \ldots dq_{3N}\, dp_2 \ldots dp_{3N}$$

with all correlations. Taking into account the long-range nature of the Coulomb interaction and applying the mean-field approximation, the BBGKY-hierarchy immediately reduces to the Vlasov equation for the single-particle probability distribution function.

Vlasov equation

Let $\psi \in f : \Gamma \sim \mathbb{R}^6 \to \mathbb{R}$ be the single-particle probability density function (for example of a beam). The number of particles dn found in the phase space volume $d^3q\, d^3p$ around the point (q, p) is, then, given by

$$dn = \psi(q, p)\, d^3q\, d^3p\,.$$

Let the accelerator Hamiltonian[a] $H \in f : \Gamma \sim \mathbb{R}^6 \to \mathbb{R}$ be given as

$$H(q, p) = H_0 + H_1$$

where H_0 stands for the single particle Hamiltonian and H_1 includes the collective effects (in mean-field approximation).
The space evolution of the single-particle probability density function of the beam under the influence of the accelerator Hamiltonian is then given by the Poisson bracket

$$\partial_s \psi = [H, \psi]\,.$$

[a]The accelerator Hamiltonian is typically transformed to generate translations in space s, denoting a location along the path of the beam, rather than in time t.

Potential well distortion

We have learned that any distribution function of the Hamiltonian is a stationary solution of the Vlasov equation. Consider the following Gaussian distribution function

$$\psi(H) \propto e^{\frac{H}{\eta\sigma_\delta^2}}$$

with the (purely longitudinal) Hamiltonian

$$H(z,\delta) = -\frac{1}{2}\eta\,\delta^2 - \frac{1}{2\eta}\left(\frac{\omega_s}{\beta c}\right)^2 z^2 + \frac{e^2}{\beta^2 EC}\int_0^z dz'' \int_{z''}^{\infty} dz'\,\rho(z')\,W_0'(z''-z')\,.$$

The distribution function can be factorized to

$$\psi(z,\delta) = e^{-\frac{\delta^2}{2\sigma_\delta^2}}\,\rho(z)$$

and we can immediately write down the equation for the stationary line-density function:

$$\rho(z) = \exp\left(-\left(\frac{\omega_s\, z}{2\eta\sigma_\delta\beta c}\right)^2 + \frac{e^2}{\eta\sigma_\delta^2\beta^2 EC}\int_0^z dz'' \int_{z''}^{\infty} dz'\,\rho(z')\,W_0'(z''-z')\right)\,.$$

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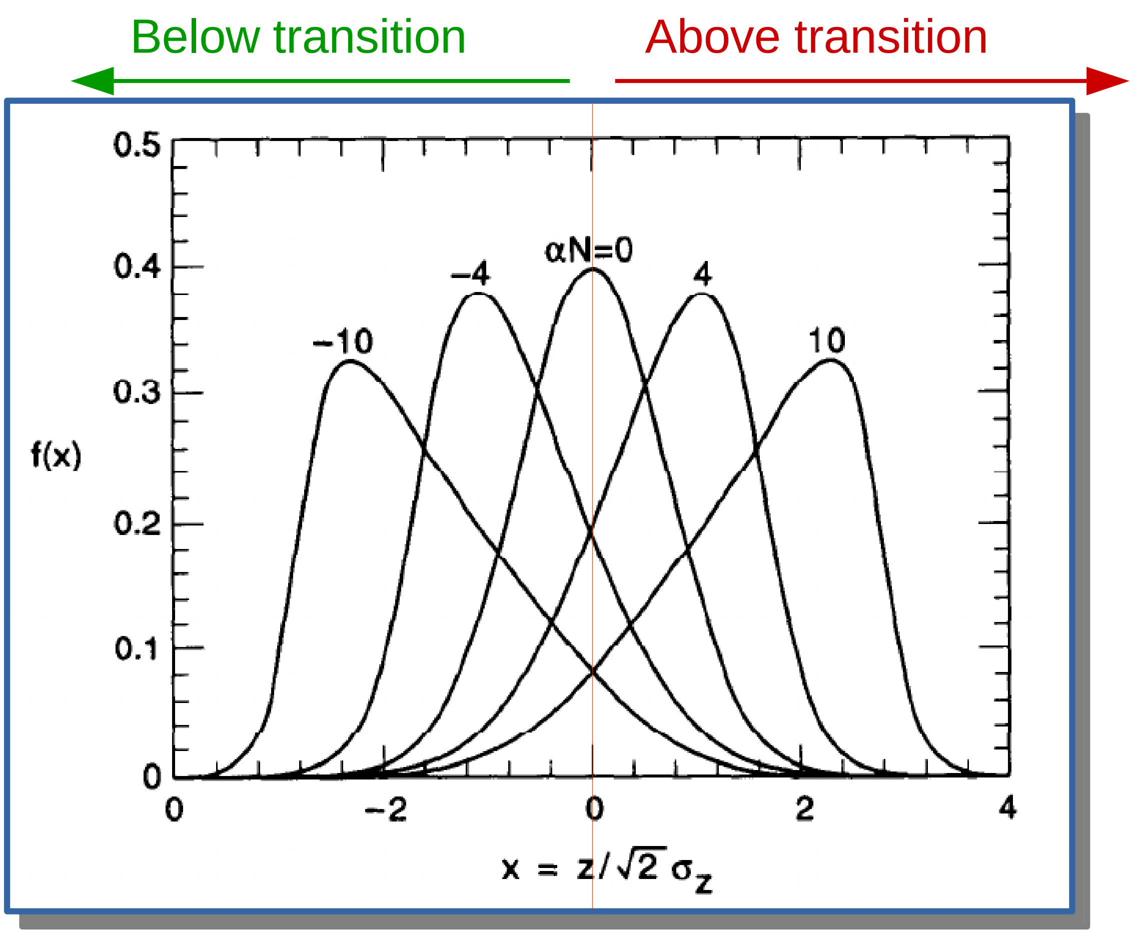
Haissinki-equation:
It is transcendental and needs to be solved numerically. The perturbed stationary solution is a result of the potential-well distortion which is the zeroth order effect from the wake fields.

$$\rho(z) = \exp\left(-\left(\frac{\omega_s\,z}{2\eta\sigma_\delta\beta c}\right)^2 + \frac{e^2}{\eta\sigma_\delta^2\beta^2 EC}\int_0^z dz'' \int_{z''}^{\infty} dz'\,\rho(z')\,W_0'(z''-z')\right)\,.$$

Potential well distortion

- From pure energy balance considerations we can already infer how the bunch will readjust in the RF bucket
- To compensate for the energy loss, the bunch will adjust the stable phase in the RF bucket – towards the tail of the bunch below transition and towards the head of the bunch above transition



The line density as a function of total bunch charge for the example of a Gaussian beam distribution and a purely resistive impedance

Synchroton tune shift due to a wake

The synchrotron tune shift due to the wakefield can be evaluated simply from an expansion of the Hamiltonian and will be simply given by the z^2-coefficient of the expansion:

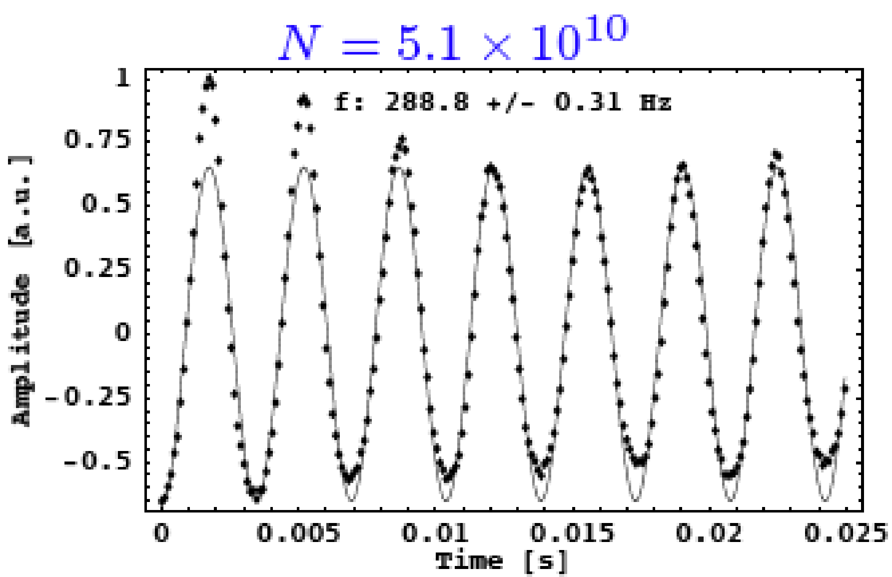
$$\begin{aligned}\Delta\omega_s &= -\frac{1}{2\omega_s}\frac{\eta e^2 c^2}{EC}\frac{\partial^2}{\partial z^2}\int dz'\,\rho(z')W_0''(z-z')\\ &= -i\frac{\eta e^2 c^2}{4\pi\omega_s EC}\int d\omega\,\hat{\rho}(\omega)\frac{\omega}{c}Z_0(\omega)\end{aligned}$$

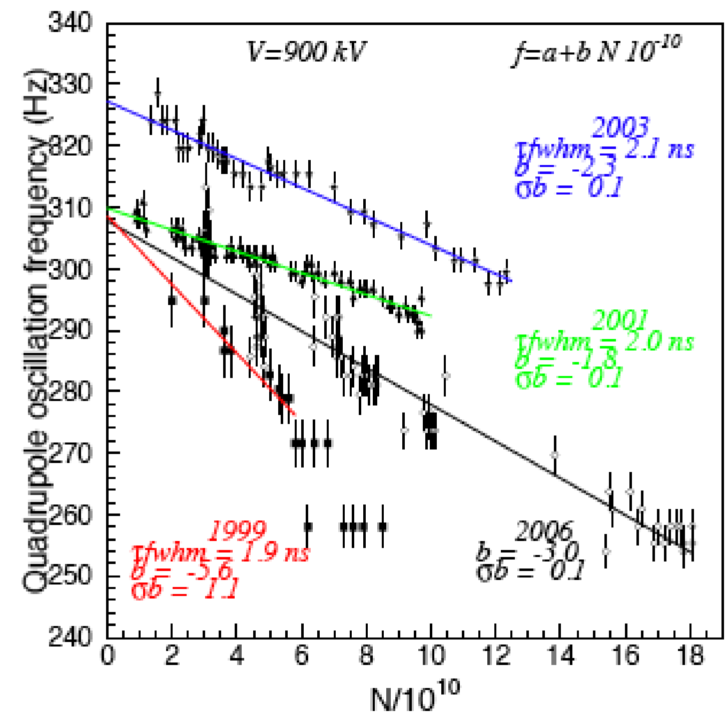
We finally arrive at the synchrotron tune shift given as

$$\Delta Q_s = -\frac{1}{4\omega_s}\frac{e^2\eta}{(2\pi)^2 E}\int d\omega\,\omega\,\hat{\rho}(\omega)\,\mathrm{Im}(Z_0(\omega))$$

SPS tune shift measurements

- The potential well leads to an intensity dependent tune shift which can be measured to probe the imaginary part of the impedance
- The technique uses the quadrupole oscillations of a bunch injected with a mismatch
- Qs can be extrapolated from bunch length or peak amplitude measurements





Evolution of the imaginary part of the machine impedance (E. Shaposhnikova, T. Bohl, J. Tuckmantel) over time

Vlasov eq. – perturbation approach

Let's now expand the single-particle probability density function ψ. We assume we have found an equilibrium distribution ψ_0 such, that

$$[H, \psi_0] = 0\,.$$

We add a small perturbation ψ_1 to this equilibrium distribution resulting in the total perturbed distribution

$$\psi = \psi_0 + \psi_1\,.$$

The time evolution of the total distribution under the accelerator Hamiltonian is given as

$$\begin{aligned}\partial_t\psi_0 + \partial_t\psi_1 &= [H_0 + H_1(\psi_0 + \psi_1), \psi_0 + \psi_1]\\ &= [H_0 + H_1(\psi_0) + H_1(\psi_1), \psi_0 + \psi_1]\,,\end{aligned}$$

where we have used that H_1 is linear in the distribution function ψ.

Vlasov eq. – perturbation approach

If we manipulate the Poisson brackets in the previous equation, we arrive to

$$\begin{aligned}\partial_t\psi_0 + \partial_t\psi_1 &= [H_0 + H_1(\psi_0), \psi_0] \\ &+ [H_0 + H_1(\psi_0), \psi_1] + [H_1(\psi_1), \psi_0] \\ &+ [H_1(\psi_1), \psi_1] \,.\end{aligned}$$

Vlasov eq. – perturbation approach

If we manipulate the Poisson brackets in the previous equation, we arrive to

$$\begin{aligned}\partial_t\psi_0 + \partial_t\psi_1 = &\left[H_0 + \boxed{H_1(\psi_0)}, \psi_0\right] \\ &+ \left[H_0 + \boxed{H_1(\psi_0)}, \psi_1\right] + \left[H_1(\psi_1), \psi_0\right] \\ &+ \left[H_1(\psi_1), \psi_1\right].\end{aligned}$$

Potential well distortion – can be absorbed in a redefinition of psi_0

Potential well perturbation – neglected as it is not essential for the mechanism of collective beam instabilities

Vlasov eq. – perturbation approach

If we manipulate the Poisson brackets in the previous equation, we arrive to

$$\begin{aligned}\partial_t\psi_0 + \partial_t\psi_1 = {} & [H_0, \psi_0] \\ & + [H_0, \psi_1] + [H_1(\psi_1), \psi_0] \\ & + [H_1(\psi_1), \psi_1] \, .\end{aligned}$$

Vlasov eq. – perturbation approach

If we manipulate the Poisson brackets in the previous equation, we arrive to

$$
\begin{aligned}
\cancel{\partial_t \psi_0} + \partial_t \psi_1 = {} & [\cancel{H_0, \psi_0}] \\
& + [H_0, \psi_1] + [H_1(\psi_1), \psi_0] \\
& + [H_1(\psi_1), \psi_1] \, .
\end{aligned}
$$

Unperturbed solution known – cancelation

Vlasov eq. – perturbation approach

If we manipulate the Poisson brackets in the previous equation, we arrive to

$$\cancel{\partial_t \psi_0} + \partial_t \psi_1 = [\cancel{H_0, \psi_0}] + [H_0, \psi_1] + [H_1(\psi_1), \psi_0] + [\cancel{H_1(\psi_1)}, \psi_1] \, .$$

Unperturbed solution known – cancelation

Second order in perturbation – neglected

Vlasov eq. – perturbation approach

We finally arrive at the Vlasov equation which expresses the time evolution of a small perturbation ψ_1 ontop of an equilibrium distribution ψ_0 due to collective effects described by the Hamiltonain $H_1(\psi_1)$

$$\partial_s \psi_1 = [H_0, \psi_1] + [H_1(\psi_1), \psi_0] \, .$$

Vlasov eq. – 1-dimensional system

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$$\partial_s \psi_1 = [H_0, \psi_1] + [H_1(\psi_1), \psi_0] \, .$$

We, then, consider the purely longitudinal Hamiltonians

$$H_0 = -\frac{1}{2}\eta\,\delta^2 - \frac{1}{2\eta}\left(\frac{\omega_s}{\beta c}\right)^2 z^2$$

$$H_1 = \frac{e^2}{\beta^2 EC}\int dz''\, V(z'')$$

$$V(z) = \int dz' \sum_{k=-\infty}^{\infty} \rho^{(0)}(z') e^{-i\Omega(s/(\beta c) - kT_0)} W_0'(z - z' - kcT_0)$$

Vlasov eq. – 1-dimensional system

We search for stationary solutions, in the broader sense, given as

$$\partial_s \psi_1 \equiv -i \frac{\Omega}{\beta c} \psi_1 \,.$$

Therefore, we can specify the solution as

$$\psi(z,\delta) = \psi_0(z,\delta) + \psi_1(z,\delta) = f_0(z,\delta) + f_1(z,\delta) e^{-i\Omega s/(\beta c)} \,.$$

Vlasov eq. – 1-dimensional system

We finally arrive at the Vlasov equation which expresses the time evolution of a small perturbation ψ_1 ontop of an equilibrium distribution ψ_0 due to collective effects described by the Hamiltonain $H_1(\psi_1)$

$$\partial_s \psi_1 = \boxed{[H_0, \psi_1]} + \boxed{[H_1(\psi_1), \psi_0]}.$$

$$\psi(z,\delta) = \psi_0(z,\delta) + \psi_1(z,\delta) = \boxed{f_0(z,\delta)} + \boxed{f_1(z,\delta)e^{-i\Omega s/(\beta c)}}.$$

We, then, consider the purely longitudinal Hamiltonians

$$\boxed{H_0 = -\frac{1}{2}\eta\,\delta^2 - \frac{1}{2\eta}\left(\frac{\omega_s}{\beta c}\right)^2 z^2}$$

$$\boxed{H_1 = \frac{e^2}{\beta^2 EC}\int dz''\,V(z)}$$

$$V(z) = \int dz' \sum_{k=-\infty}^{\infty} \rho^{(0)}(z')e^{-i\Omega(s/(\beta c)-kT_0)}W_0'(z-z'-kcT_0)$$

We now need to evaluate the two Poisson brackets using the distribution functions and the Hamiltonians we have developed

Vlasov eq. – 1-dimensional system

Step #1: evaluate first Poisson bracket

Given the form of the Hamiltonian H_0 it will be helpful to move to polar coordinates

$$z = r\cos\phi\,, \qquad r = \sqrt{z^2 + \left(\frac{\eta\beta c}{\omega_s}\right)^2 \delta^2}\,, \qquad \frac{\partial}{\partial z} = \frac{\partial r}{\partial z}\frac{\partial}{\partial r} + \frac{\partial \phi}{\partial z}\frac{\partial}{\partial \phi}$$

$$\delta = \frac{\omega_s}{\eta\beta c} r\sin\phi\,, \qquad \phi = \arctan\left(\frac{\eta\beta c}{\omega_s}\frac{\delta}{z}\right)\,, \qquad \frac{\partial}{\partial \delta} = \frac{\partial r}{\partial \delta}\frac{\partial}{\partial r} + \frac{\partial \phi}{\partial \delta}\frac{\partial}{\partial \phi}$$

Vlasov eq. – 1-dimensional system

Step #1: evaluate first Poisson bracket

The Poisson bracket becomes:

$$[H_0, \psi_1] = \left(\eta\, \delta \frac{\partial f_1}{\partial z} - \left(\frac{\omega_s}{\beta c} \right)^2 \frac{z}{\eta} \frac{\partial f_1}{\partial \delta} \right) e^{-i\Omega s/(\beta c)}$$

$$= -\frac{\omega_s}{\beta c} \frac{\partial f_1}{\partial \phi} e^{-i\Omega s/(\beta c)}$$

Vlasov eq. – 1-dimensional system

Step #2: evaluate second Poisson bracket

The Poisson bracket becomes:

$$[H_0, \psi_1] = \left(\eta\, \delta \frac{\partial f_1}{\partial z} - \left(\frac{\omega_s}{\beta c} \right)^2 \frac{z}{\eta} \frac{\partial f_1}{\partial \delta} \right) e^{-i\Omega s/(\beta c)}$$

$$= -\frac{\omega_s}{\beta c} \frac{\partial f_1}{\partial \phi} e^{-i\Omega s/(\beta c)}$$

Because $[H_0, f_0] = 0$ ($[H_0, \psi_0] = 0$) it follows that $f_0(z, \delta) = f_0(r)$. Then, the second Poisson bracket evaluates to

$$[H_1, \psi_0] = \frac{\partial H_1}{\partial z} \frac{\partial f_0}{\partial \delta}$$

$$= \frac{e^2}{\beta^2 E C} \frac{\eta \beta c}{\omega_s} \sin \phi\, f_0'\, V(z)$$

Vlasov eq. – 1-dimensional system

Step #2: evaluate second Poisson bracket

We then write the Vlasov equation with the evaluated Poisson brackets as

$$
\begin{aligned}
-i\frac{\Omega}{\beta c} f_1 e^{-i\Omega s/(\beta c)} &= -\frac{\omega_s}{\beta c}\frac{\partial f_1}{\partial \phi} e^{-i\Omega s/(\beta c)} + \frac{e^2}{\beta^2 EC}\frac{\eta\beta c}{\omega_s}\sin\phi\, f_0'\, V(z) \\
&= -\frac{\omega_s}{\beta c}\frac{\partial f_1}{\partial \phi} e^{-i\Omega s/(\beta c)} + \frac{e^2}{\beta^2 EC}\frac{\eta\beta c}{\omega_s}\sin\phi\, f_0' \\
&\times \int dz' \sum_{k=-\infty}^{\infty} \rho(z') e^{-i\Omega(s/(\beta c)-kT_0)} W_0'(z - z' - kcT_0)
\end{aligned}
$$

Vlasov eq. – 1-dimensional system

Step #3a: Φ decomposition of f1

We now need to find appropriate decompositions for f_1. In a straightforward and general approach we opt for the Fourier transform:

ϕ-decomposition:

$$f_1(r, \phi) = \sum_l a_l R_l(r) e^{il\phi} \, .$$

Vlasov eq. – 1-dimensional system

Step #3a: Φ decomposition of f1

Inserting the decomposition above we arrive at the Vlasov equation

$$\sum_{l''} a_{l''} R_{l''}(r) e^{il''\phi} (\Omega - l''\omega_s) e^{-i\Omega s/(\beta c)} = i \frac{e^2}{EC} \frac{\eta c^2}{\omega_s} \sin\phi\, f_0'$$

$$\times \int dz' \sum_{k=-\infty}^{\infty} \rho(z') e^{-i\Omega(s/(\beta c) - kT_0)} W_0'(z - z' - kcT_0)$$

Vlasov eq. – 1-dimensional system

Step #3b: frequency domain

Inserting the decomposition above we arrive at the Vlasov equation

$$\sum_{l''} a_{l''} R_{l''}(r) e^{il''\phi} (\Omega - l''\omega_s) e^{-i\Omega s/(\beta c)} = i \frac{e^2}{EC} \frac{\eta c^2}{\omega_s} \sin\phi \, f_0'$$

$$\times \underbrace{\int dz' \sum_{k=-\infty}^{\infty} \rho(z') e^{-i\Omega(s/(\beta c) - kT_0)} W_0'(z - z' - kcT_0)}_{\text{Frequency domain}}$$

Vlasov eq. – 1-dimensional system

Step #3b: frequency domain

Inserting the decomposition above we arrive at the Vlasov equation

$$\sum_{l''} a_{l''} R_{l''}(r) e^{il''\phi} (\Omega - l''\omega_s) e^{-i\Omega s/(\beta c)} = i \frac{e^2}{EC} \frac{\eta c^2}{\omega_s} \sin\phi \, f_0'$$

$$\times \int dz' \sum_{k=-\infty}^{\infty} \rho(z') e^{-i\Omega(s/(\beta c) - kT_0)} W_0'(z - z' - kcT_0)$$

Vlasov eq. – 1-dimensional system

Step #3b: frequency domain

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$$\sum_{l''} a_{l''} R_{l''}(r) e^{il''\phi} (\Omega - l''\omega_s) e^{-i\Omega s/(\beta c)} = i \frac{e^2}{EC} \frac{\eta c^2}{\omega_s} \sin\phi \, f_0'$$

$$\times \int dz' \sum_{k=-\infty}^{\infty} \frac{1}{2\pi\beta c} \int d\omega \, \hat{\rho}(\omega) e^{i\omega z'/(\beta c)}$$

$$\times \frac{1}{2\pi} \int d\omega' \, e^{i\omega'(z-z'-k\beta c T_0)/(\beta c)} Z(\omega') \, e^{-i\Omega(s/(\beta c) - kT_0)}$$

Vlasov eq. – 1-dimensional system

Step #3b: frequency domain

Inserting the decomposition above we arrive at the Vlasov equation

$$\sum_{l''} a_{l''} R_{l''}(r) e^{il''\phi} (\Omega - l''\omega_s) e^{-i\Omega s/(\beta c)} = i \frac{e^2}{EC} \frac{\eta c^2}{\omega_s} \sin\phi\, f_0'$$

$$\times \frac{1}{4\pi^2 \beta c} \sum_{k=-\infty}^{\infty} \int d\omega\, d\omega'\, dz'\, \hat{\rho}(\omega) e^{i\omega' z/(\beta c)} Z(\omega')$$

$$\times\, e^{i(\omega-\omega')z'/(\beta c)} e^{ikT_0(\Omega-\omega')} e^{-i\Omega s/(\beta c)}$$

Vlasov eq. – 1-dimensional system

Step #3b: frequency domain

Inserting the decomposition above we arrive at the Vlasov equation

$$
\begin{aligned}
\sum_{l''} a_{l''} R_{l''}(r) e^{il''\phi}(\Omega - l''\omega_s) e^{-i\Omega s/(\beta c)} &= i\frac{e^2}{EC}\frac{\eta c^2}{\omega_s}\sin\phi\, f_0' \\
&\times \frac{1}{4\pi^2\beta c}\sum_{k=-\infty}^{\infty}\int d\omega\, d\omega'\, dz'\, \hat{\rho}(\omega) e^{i\omega' z/(\beta c)} Z(\omega') \\
&\times \underbrace{e^{i(\omega-\omega')z'/(\beta c)} e^{ikT_0(\Omega-\omega')} e^{-i\Omega s/(\beta c)}}_{\frac{1}{2\pi}\int dk\, e^{ik(x-x')} = \delta(x-x')}
\end{aligned}
$$

Vlasov eq. – 1-dimensional system

Step #3b: frequency domain

Inserting the decomposition above we arrive at the Vlasov equation

$$\sum_{l''} a_{l''} R_{l''}(r) e^{il''\phi} (\Omega - l''\omega_s) e^{-i\Omega s/(\beta c)} = i \frac{e^2}{EC} \frac{\eta c^2}{\omega_s} \sin\phi \, f_0'$$

$$\times \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \int d\omega \, \hat{\rho}(\omega) e^{i\omega z/(\beta c)} Z(\omega) e^{ikT_0(\Omega-\omega)} e^{-i\Omega s/(\beta c)}$$

Vlasov eq. – 1-dimensional system

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$$\sum_{l''} a_{l''} R_{l''}(r) e^{il''\phi} (\Omega - l''\omega_s) e^{-i\Omega s/(\beta c)} = i \frac{e^2}{EC} \frac{\eta c^2}{\omega_s} \sin\phi \, f_0'$$

$$\times \underbrace{\frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \int d\omega \, \hat{\rho}(\omega) e^{i\omega z/(\beta c)} Z(\omega) e^{ikT_0(\Omega-\omega)} e^{-i\Omega s/(\beta c)}}$$

$$\sum_{k=-\infty}^{\infty} e^{ikx} = 2\pi \sum_{p=-\infty}^{\infty} \delta(x - 2\pi p)$$

Vlasov eq. – 1-dimensional system

Step #3b: frequency domain

Inserting the decomposition above we arrive at the Vlasov equation

$$\sum_{l''} a_{l''} R_{l''}(r) e^{il''\phi}(\Omega - l''\omega_s) e^{-i\Omega s/(\beta c)} = i\frac{e^2}{EC}\frac{\eta c^2}{\omega_s}\sin\phi\, f_0'$$

$$\times \frac{1}{T_0}\sum_{p=-\infty}^{\infty} \hat{\rho}(\Omega - p\omega_0) e^{i(\Omega - p\omega_0)z/(\beta c)} Z(\Omega - p\omega_0) e^{-i\Omega s/(\beta c)}\,,$$

and we finally obtain

$$\sum_{l''} a_{l''} R_{l''}(r) e^{il''\phi}(\Omega - l''\omega_s) = i\frac{e^2}{\beta E T_0^2}\frac{\eta c}{\omega_s}\sin\phi\, f_0'$$

$$\times \sum_{p=-\infty}^{\infty} \hat{\rho}(\Omega - p\omega_0) e^{i(\Omega - p\omega_0)z/(\beta c)} Z(\Omega - p\omega_0)\,.$$

Vlasov eq. – 1-dimensional system

Step #3b: frequency domain

Some standard algrebra (multiply, integrate and use orthonormality) immediately yields:

$$(\Omega - l\omega_s)a_l R_l(r) = i\frac{e^2}{\beta E T_0^2}\frac{\eta c}{\omega_s} f_0' \sum_{p=-\infty}^{\infty} \int d\phi\, \sin\phi\, e^{-il\phi + i\omega'(r\cos\phi)/(\beta c)} \hat{\rho}(\omega') Z(\omega')$$

Vlasov eq. – 1-dimensional system

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Vlasov eq. – 1-dimensional system

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Next, we perform the "inverse projection" of $\hat{\rho}(\omega')$ (details in Chao eq. 6.75):

Vlasov eq. – 1-dimensional system

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$$(\Omega - l\omega_s)a_l R_l(r) = -2\pi i\beta c \frac{e^2}{E T_0^2} l \frac{f_0'}{r} \sum_{l'} \int r' \, dr' \, a_{l'} R_{l'}(r') i^{l-l'}$$
$$\times \sum_{p=-\infty}^{\infty} J_l\left(\frac{\omega' r}{\beta c}\right) \frac{Z(\omega')}{\omega'} J_{l'}\left(\frac{\omega' r'}{\beta c}\right)$$

Vlasov eq. – 1-dimensional system

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$$\times \sum_{p=-\infty}^{\infty} J_l\left(\frac{\omega' r}{\beta c}\right) \frac{Z(\omega')}{\omega'} J_{l'}\left(\frac{\omega' r'}{\beta c}\right)$$

We now need to find appropriate decompositions for $R_l(r)$:
r-decomposition

$$R_l(r) = W(r) \sum_k b_{kl} u_{kl}(r)$$

$$\int r\, dr\, W(r) u_{kl}(r) u_{k'l'}(r) = \delta_{kk'}\delta_{ll'}$$

Vlasov eq. – 1-dimensional system

Step #3c: r decomposition of f1

Before inserting our decomposition, we define ourselves the weight function $W(r)$ as

$$W(r) = -\frac{\omega_s}{N\eta\beta c}\frac{f_0'(r)}{r}\,.$$

All decompositions and orthonormality conditions from the previous slide must then be selected with respect to this weight function. In practice, this is a non-trivial task but, for now, we will pragmatically assume that this can be done.
Inserting the r-decomposition, with the weight function as defined above, we obtain:

$$\begin{aligned}(\Omega - l\omega_s)a_l W(r)\sum_{k''} b_{k''l}u_{k''l}(r) &= 2\pi i\frac{N\eta\beta^2c^2}{\omega_s}\frac{e^2}{ET_0^2}\,l\,W(r)\\ &\times\sum_{k'}\sum_{l'}\int r'\,dr'\,a_{l'}W(r')b_{k'l'}u_{k'l'}(r')i^{l-l'}\\ &\times\sum_{p=-\infty}^{\infty}J_l\left(\frac{\omega' r}{\beta c}\right)\frac{Z(\omega')}{\omega'}J_{l'}\left(\frac{\omega' r'}{\beta c}\right)\end{aligned}$$

Vlasov eq. – 1-dimensional system

Step #3c: r decomposition of f1

Again, by standard algebraic manipulations, we multiply and integrate by $\int r\,dr\,u_{kl}(r)$. Making use of the orthonormality conditions we arrive at

$$
\begin{aligned}
(\Omega - l\omega_s)a_l\,b_{kl} = 2\pi i \frac{N\eta\beta^2c^2}{\omega_s}\frac{e^2}{ET_0^2}\,l \sum_{p=-\infty}^{\infty}\sum_{k'}\sum_{l'} a_{l'}\,b_{k'l'}\,i^{l-l'} \\
\times \int r\,dr\,W(r)u_{kl}(r)J_l\left(\frac{\omega' r}{\beta c}\right) \\
\times \frac{Z(\omega')}{\omega'} \\
\times \int r'\,dr'\,W(r')u_{k'l'}(r')J_{l'}\left(\frac{\omega' r'}{\beta c}\right)
\end{aligned}
$$

Vlasov eq. – 1-dimensional system

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\times \underbrace{\int r\, dr\, W(r) u_{kl}(r) J_l\left(\frac{\omega' r}{\beta c}\right)}_{v_{kl}(\omega')} \\
\times \frac{Z(\omega')}{\omega'} \\
\times \underbrace{\int r'\, dr'\, W(r') u_{k'l'}(r') J_{l'}\left(\frac{\omega' r'}{\beta c}\right)}_{v_{k'l'}(\omega')}
\end{aligned}
$$

Vlasov eq. – 1-dimensional system

Step #4: formulate eigenvalue problem

We write the previous equation as

$$(\Omega - l\omega_s) a_l \, b_{kl} = 2\pi i \frac{N\eta\beta^2c^2}{\omega_s} \frac{e^2}{ET_0^2} \, l \sum_{k'} \sum_{l'} a_{l'} \, b_{k'l'} \, i^{l-l'}$$
$$\times \sum_{p=-\infty}^{\infty} v_{kl}(\omega') \, \frac{Z(\omega')}{\omega'} \, v_{k'l'}(\omega')$$
$$= \sum_{k'} \sum_{l'} a_{l'} \, b_{k'l'} \, \mathcal{M}_{kk',ll'}$$

with the interation matrix $\mathcal{M}_{kk',ll'}$ given as

$$\mathcal{M}_{kk',ll'} = 2\pi i \frac{N\eta\beta^2c^2}{\omega_s} \frac{e^2}{ET_0^2} \, l \, i^{l-l'} \sum_{p=-\infty}^{\infty} v_{kl}(\omega') \, \frac{Z(\omega')}{\omega'} \, v_{k'l'}(\omega')$$

Vlasov eq. – 1-dimensional system

Step #4: formulate eigenvalue problem

We have finally arrived at a linear set of equations

$$(\Omega - l\omega_s)a_l\, b_{kl} = \sum_{k'}\sum_{l'} a_{l'}\, b_{k'l'}\, \mathcal{M}_{kk',ll'}\,.$$

With

$$M_{kk',ll'} = l\omega_s\delta_{kk'}\delta_{ll'} + \mathcal{M}_{kk',ll'}\,,$$

this can be written as

$$\Big(\Omega\,\mathbf{1} - M\Big)\, v = 0\,,$$

a classical eigenvalue problem. We must, therefore, diagonalise the matrix M by solving the secular equation

$$\det\Big(\Omega\,\mathbf{1} - M\Big) = 0$$

to find the eigenvalues and the corresponding eigenvectors.

Discussion of the EV problem

For a non-trivial solution to exist, the eigenvalues must satisfy

$$\det\Big(\left(\Omega - l\omega_s\right)\delta_{kk'}\delta_{ll'} - \mathcal{M}_{kk',ll'}\Big) = 0$$

So how do we solve a stability problem? The steps will always be the same:

- Typically, we will start off from a particle distribution function together with an impedance.
- We construct our weight function $W(r)$ from the unperturbed stationary solution $f_0(r)$ and find the corresponding basis functions $u_{kl}(r)$ which we can then use to compute $v_{kl}(r)$ and the interaction matrix $\mathcal{M}_{kk',ll'}$.
- If for a given choice of basis functions $u_{kl}(r)$ the interaction matrix turns out to be diagonal, the problem is readily solved. Otherwise, the matrix needs to be diagonalised. This will yield the eigenvalues and eigenvectors for each k, l-mode.

A simple picture of the modes

Let's assume an airbag distribution by assuming a weight function

$$W(r) = \frac{1}{\pi \hat{z}^3} \delta(r - \hat{z}) \,,$$

so the radial part of the perturbation becomes

$$R_l(r) \propto \delta(r - \hat{z}) \,.$$

There are infinite azimuthal modes, each mode resembling a particular stationary oscillation. Assuming for now $N = 0$, the zero intensity limit, it follows that the eingenvalue is readily obtained as

$$\Omega = l\omega_s$$

and we can immediately write down the perturbation as

$$\psi_1(r, \phi) \propto \delta(r - \hat{z}) e^{il\phi} e^{-il\omega_s s/(\beta c)}$$

A simple picture of the modes

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$$\ldots \frac{1}{\hat{z}^3}\delta(r-\hat{z})\,,$$

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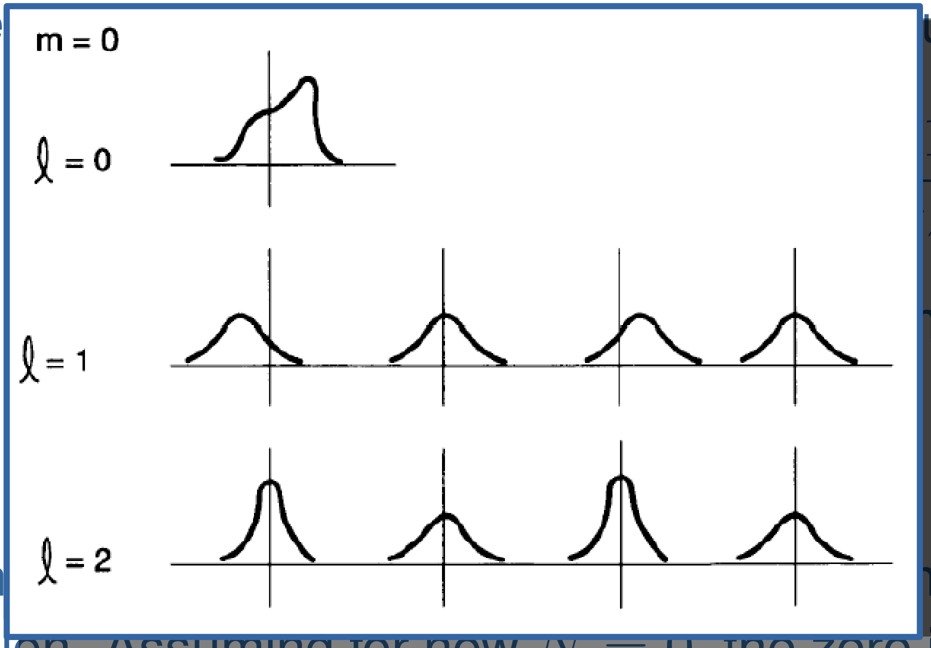

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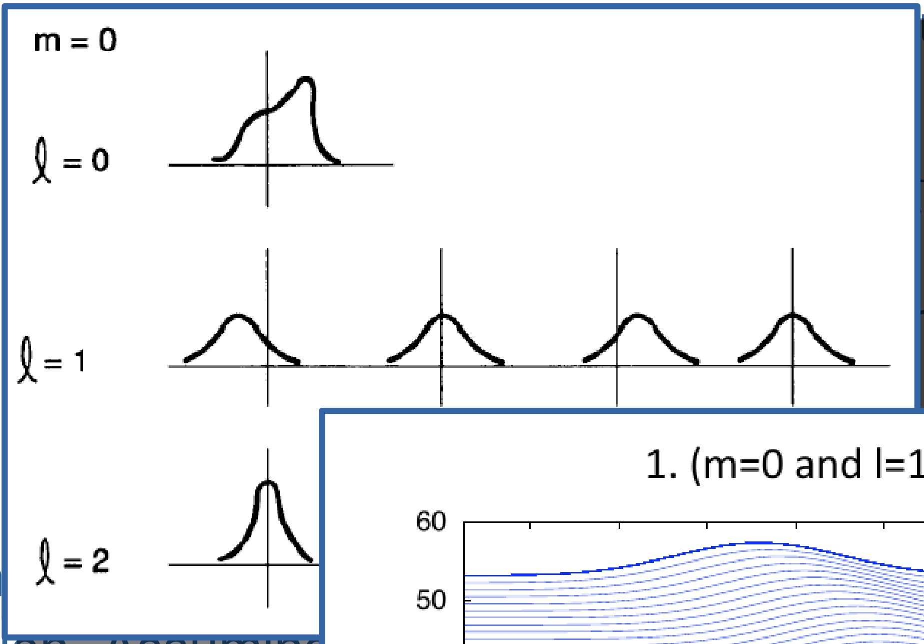

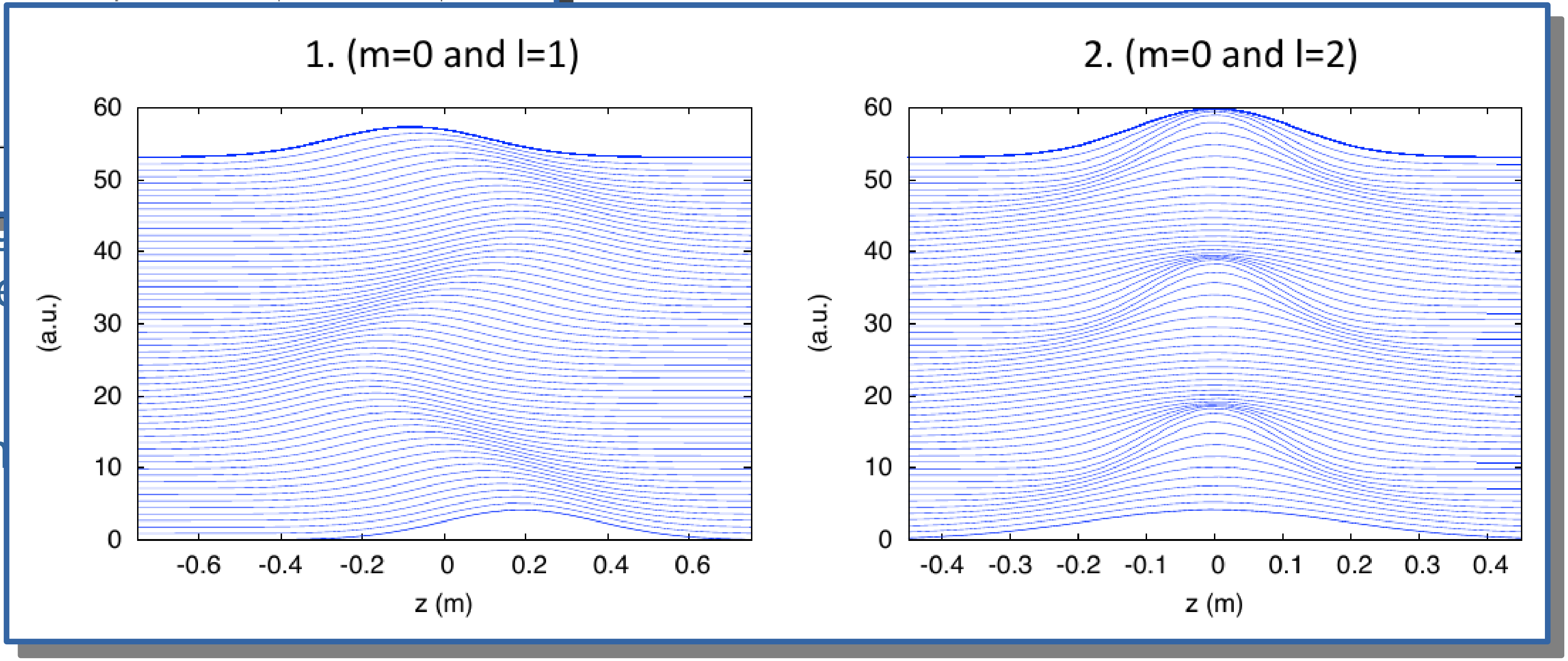

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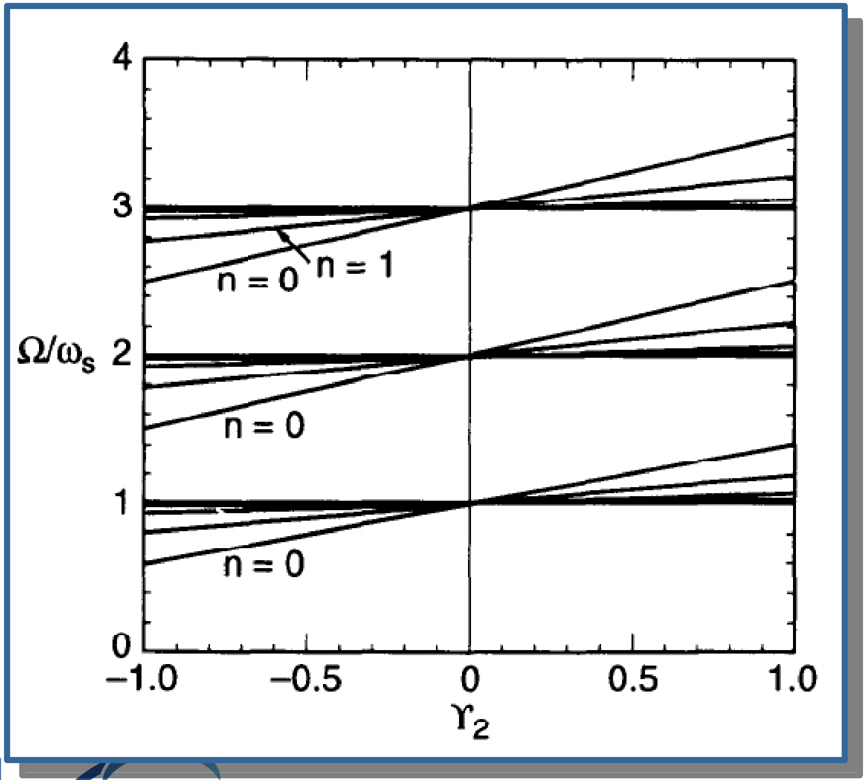
Tune shift and instabilities

Let's take a look at the general structure of the eigenvalue problem and try to identify some particular cases. We express the eigenvalue problem as

$$\left(\frac{\Omega}{\omega_s} - l\right)\delta_{kk',ll'} = 2\pi i \frac{N\eta\beta^2c^2}{\omega_s}\frac{e^2}{ET_0^2}\, l\, i^{l-l'} \sum_{p=-\infty}^{\infty} v_{kl}(\omega')\frac{Z(\omega')}{\omega'} v_{k'l'}(\omega')$$

$$\hat{\rho}_l(\omega) \propto i^{-l} v_{kl}(\omega)$$



- For low intensities, we can fix the azimuthal mode number l
- Each mode l will have rays of radial modes shifting linearly with the intensity

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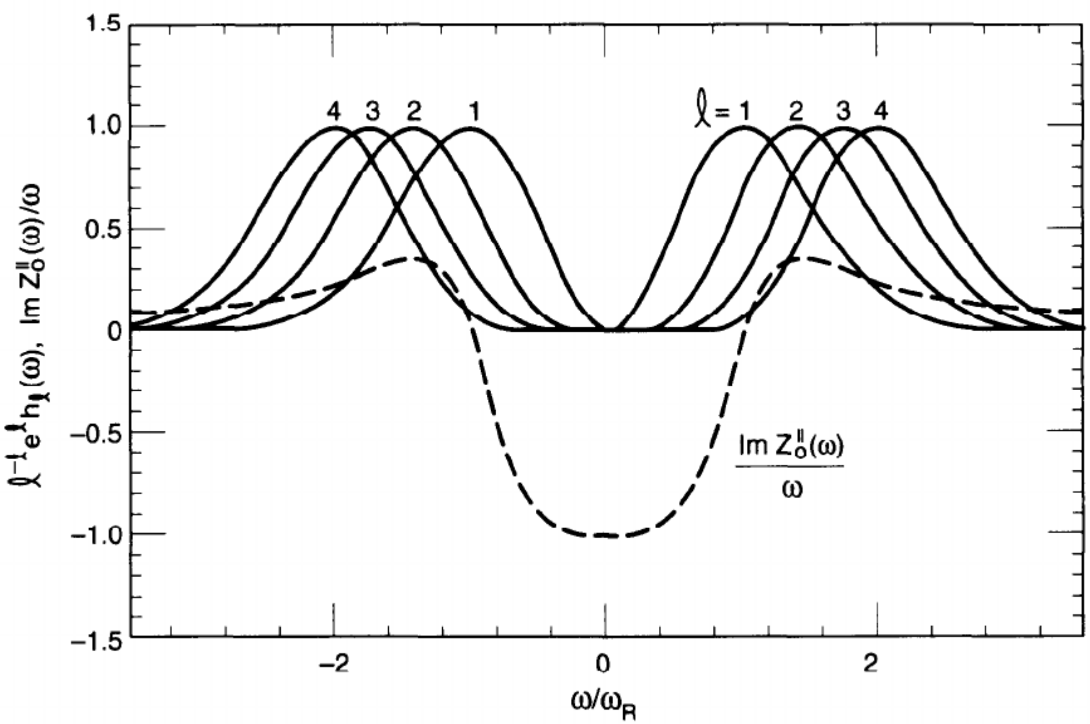

- For low intensities, we can fix the azimuthal mode number l
- Each mode l will have rays of radial modes shifting linearly with the intensity
- The tune shift is given by the overlap integral of the imaginary part of the impedance with the mode power spectrum.

Tune shift and instabilities

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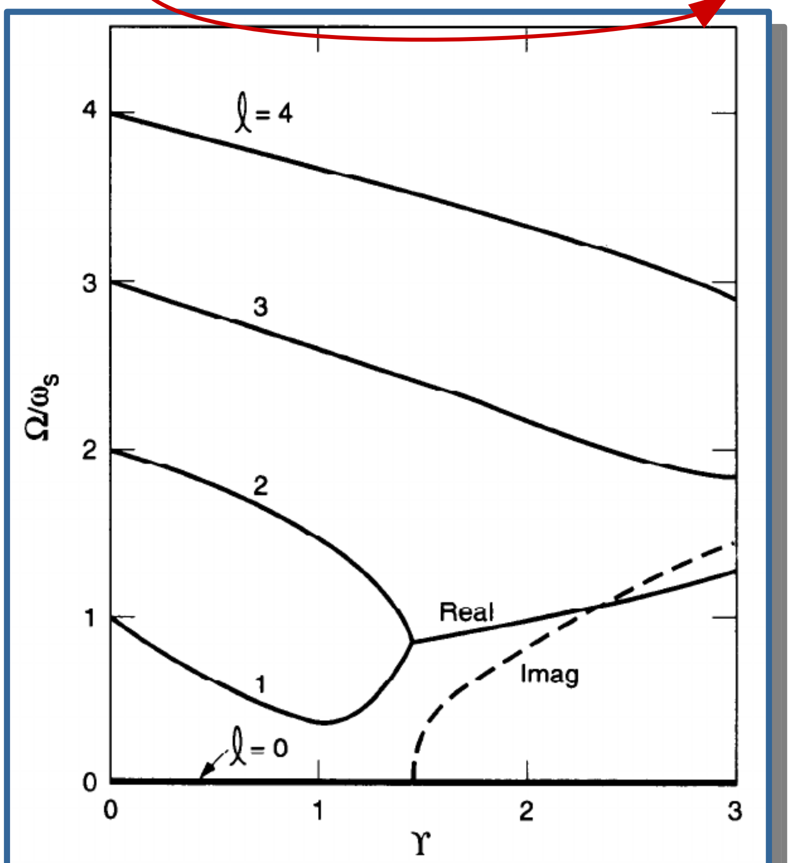
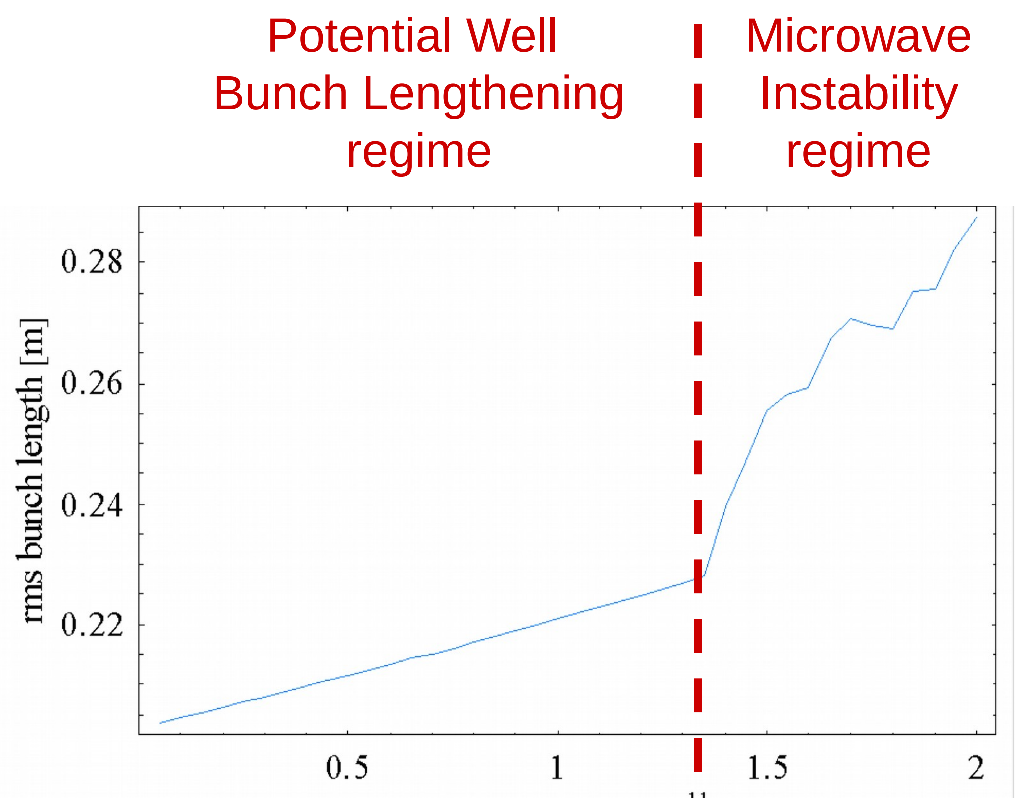


- For high intensities, azimuthal modes can couple. The azimuthal mode number can no longer be treatet as a constant but becomes part of the interaction matrix
- The tune shift with intensity is no longer linear. When the azimuthal modes couple, the eigenvalues become imaginary and the beam becomes unstable. This is know as the mode coupling or microwave or turbulent instability

Instabilities measurements

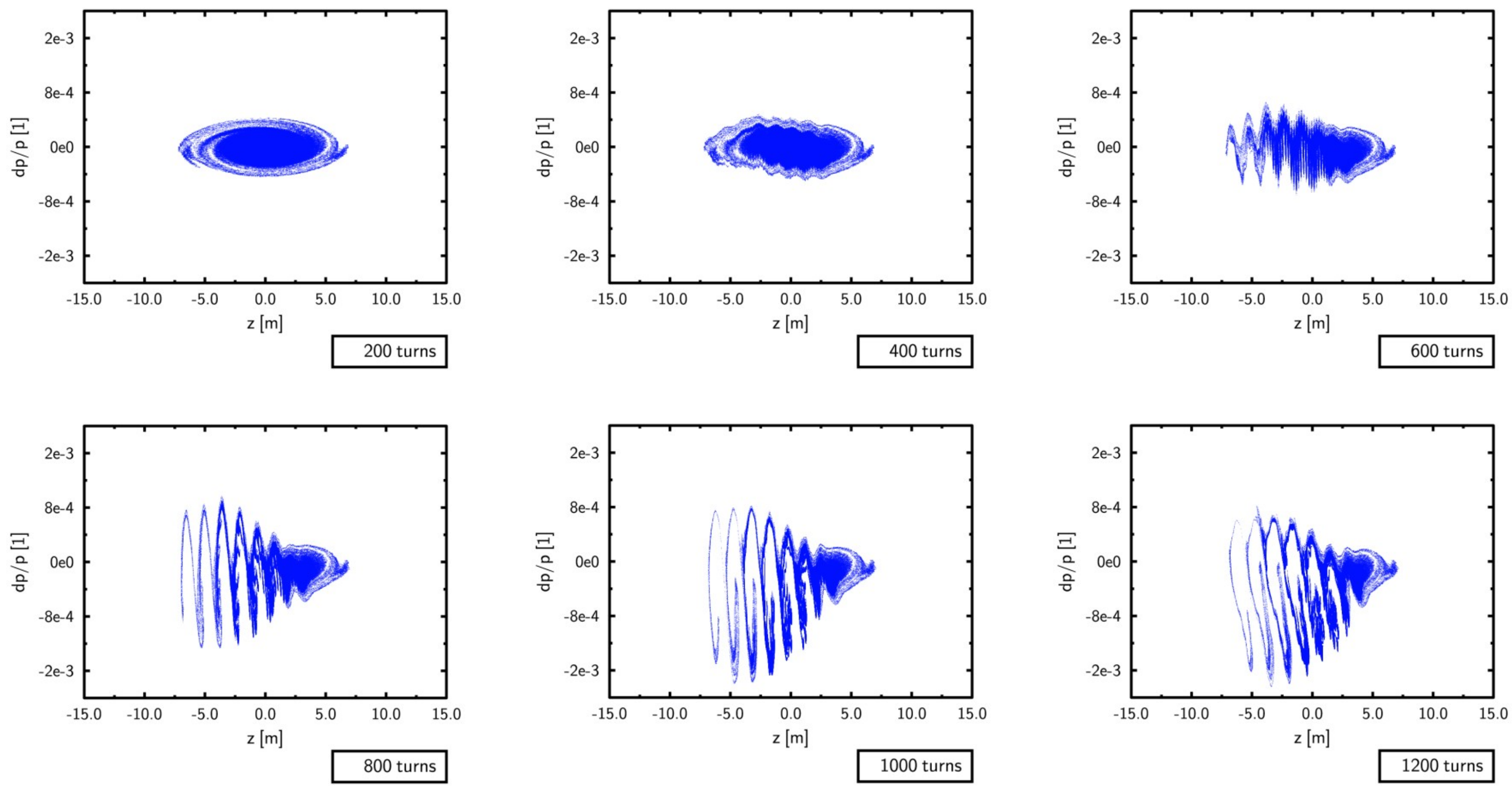
- Initialising a matched bunch at low intensity, two regimes are found
- Bunch lengthening/emittance blow up regime with linear increase of the bunch length as a function of the bunch intensity
- Unstable regime (turbulent bunch lengthening)



Mode coupling threshold which we will discuss after having introduced the transvers plane

Instabilities measurements

- Examples of numerical simulations of a debunching bunch with an SPS impedance model
- The microwave instability on a debunching bunch is used at the SPS for probing the machine impedance (E. Shaposhnikova, T. Bohl, H. Timkó, et al.)
- Spectrum of bunch profile reveals important components for the impedance



Some remarks on the EV problem

- Before studying some consequence of the EV problem, we will now add the transverse plane, thus, moving to the 2-dimensional problem.

Summary

- We have reviewed some basic Hamilton mechanics and obained the Vlasov equation
- We have introduced a collective part of the Hamiltonian that describes the interaction with wakefields
- This directly lead to the potential well distortion of the stationary solution
- We have introduced a perturbation on the stationary solution and used the Vlasov equation to obtain a system of equations that describes the time evolution of this perturbation
- After a lengthy set of algebraic manipulations we derived the interaction matrix and its implications on the complex tuneshift
  - We identified intrabunch modes
  - We identified how modes interact with impedances to generate tune shifts
  - We identified how modes can couple to generate instabilities

THE END