Perturbation Formalism

What we have learned yersterday

- A multi-particle system is well described by the single particle probability density function $\psi(q,p)$.
- Given a pdf $\psi(q,p)$ and a Hamiltonian $H(q,p)$, the evolution of ψ is given by the Poisson bracket $\partial_s \psi = [H, \psi]$.
- The zeroth order effect of the collective term of the Hamiltonian leads to a stationary distortion of the unperturbed stationar distribution. This is the potential well distortion.
- The remaining part of the collective term leads to a complex tune shift of the coherent mode described by the interaction matrix.
- The value of the complex tune shift together with the associated collective modes is obtained from the diagonalisation of the interaction matrix.
- Two regimes could be identified. The potential well distortion leads to a stationary bunch shortening or bunch lengthening. The turbulent regime leads to instabilitiers.

Some remarks on the EV problem

- Before studying some consequence of the EV problem, we will now add the transverse plane, thus, moving to the 2-dimensional problem.

Vlasov eq. – 2-dimensional system

We immediately start with the Vlasov equation which expresses the time evolution of a small perturbation ψ_1 ontop of an equilibrium distribution ψ_0 due to collective effects described by the Hamiltonain $H_1(\psi_1)$

$$\partial_s \psi_1 = [H_0, \psi_1] + [H_1(\psi_1), \psi_0] \,.$$

We, then, consider the combined transverse and longitudinal Hamiltonians[a]

$$H_0 = \underbrace{\frac{1}{2} p_y^2 + \left(\frac{Q_y}{R}\right)^2 y^2}_{H_\perp} \underbrace{- \frac{1}{2} \eta \, \delta^2 - \frac{1}{2\eta} \left(\frac{\omega_s}{\beta c}\right)^2 z^2}_{H_\parallel}$$

$$H_1^{(mn)} = \frac{e^2}{\beta^2 EC} \int dy \, \frac{y^n}{n!} V_m(z)$$

$$V_{mn}(z) = \int dz' \sum_{k=-\infty}^{\infty} \rho^{(m)}(z') e^{-i\Omega(s/(\beta c) - kT_0)} W_{mn}(z - z' - kcT_0)$$

[a] We focus here on one of the two transverse planes only. The second will be equivalent in its treatment.

Vlasov eq. – 2-dimensional system

Again, we search for stationary solutions, in the broader sense, given as

$$\partial_s \psi_1 \equiv -i \frac{\Omega}{\beta c} \psi_1 \, .$$

Furthermore, with the Hamiltonians satifying the relations

- $[H_\perp, H_{||}] = 0$

we can specify the solution as

$$\begin{aligned}
\psi(y, p_y, z, \delta) &= \psi_0(y, p_y, z, \delta) + \psi_1(y, p_y, z, \delta) \\
&= h_0(y, p_y, z, \delta) + h_1(y, p_y, z, \delta) e^{-i\Omega s/(\beta c)} \\
&= g_0(y, p_y) f_0(z, \delta) + g_1(y, p_y) f_1(z, \delta) e^{-i\Omega s/(\beta c)} \, .
\end{aligned}$$

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$$\partial_s \psi_1 = \boxed{[H_0, \psi_1]} + \boxed{[H_1(\psi_1), \psi_0]}.$$

$$\psi(y, p_y, z, \delta) = \boxed{g_0(y, p_y) f_0(z, \delta)} + \boxed{g_1(y, p_y) f_1(z, \delta) e^{-i\Omega s/(\beta c)}}$$

We, then, consider the combined transverse and longitudinal Hamiltonians

$$\boxed{H_0 = \frac{1}{2} p_y^2 + \left(\frac{Q_y}{R}\right)^2 y^2 - \frac{1}{2}\eta\,\delta^2 - \frac{1}{2\eta}\left(\frac{\omega_s}{\beta c}\right)^2 z^2}$$

$$\boxed{H_1^{(mn)} = \frac{e^2}{\beta^2 E C} \int dy\, \frac{y^n}{n!} V_{mn}(z)}$$

We now need to evaluate the two Poisson brackets using the distribution functions and the Hamiltonians we have developed

$$V_{mn}(z) = \int dz' \sum_{k=-\infty}^{\infty} \rho^{(m)}(z') e^{-i\Omega(s/(\beta c) - kT_0)} W_{mn}(z - z' - kcT_0)$$

Vlasov eq. – 2-dimensional system

Step #1: evaluate first Poisson bracket

Given the form of the Hamiltonians $H_\perp$ and $H_\|$, it will be helpful to move to action-angle variables and to polar coordinates, respectively:

$$y = \sqrt{\frac{2J_yR}{Q_y}}\cos\theta\,, \qquad J_y = \frac{1}{2}\left(\frac{Q_y}{R}\,y^2 + \frac{R}{Q_y}\,p_y^2\right)\,, \qquad \frac{\partial}{\partial y} = \frac{\partial J_y}{\partial y}\frac{\partial}{\partial J_y} + \frac{\partial\theta}{\partial y}\frac{\partial}{\partial\theta}$$

$$p_y = -\sqrt{\frac{2J_yQ_y}{R}}\sin\theta\,, \qquad \phi = \arctan\left(-\frac{R}{Q_y}\frac{p_y}{y}\right)\,, \qquad \frac{\partial}{\partial p_y} = \frac{\partial J_y}{\partial p_y}\frac{\partial}{\partial J_y} + \frac{\partial\theta}{\partial p_y}\frac{\partial}{\partial\theta}$$

$$z = r\cos\phi\,, \qquad r = \sqrt{z^2 + \left(\frac{\eta\beta c}{\omega_s}\right)^2\delta^2}\,, \qquad \frac{\partial}{\partial z} = \frac{\partial r}{\partial z}\frac{\partial}{\partial r} + \frac{\partial\phi}{\partial z}\frac{\partial}{\partial\phi}$$

$$\delta = \frac{\omega_s}{\eta\beta c}r\sin\phi\,, \qquad \phi = \arctan\left(\frac{\eta\beta c}{\omega_s}\frac{\delta}{z}\right)\,, \qquad \frac{\partial}{\partial\delta} = \frac{\partial r}{\partial\delta}\frac{\partial}{\partial r} + \frac{\partial\phi}{\partial\delta}\frac{\partial}{\partial\phi}$$

Vlasov eq. – 2-dimensional system

Step #1: evaluate first Poisson bracket

The Poisson bracket becomes:

$$
\begin{aligned}
[H_0, \psi_1] &= \left(f_1 \left(\left(\frac{Q_y}{R} \right)^2 y \frac{\partial g_1}{\partial p_y} - p_y \frac{\partial g_1}{\partial y} \right) + \left(g_1 \left(\eta\, \delta \frac{\partial f_1}{\partial z} - \left(\frac{\omega_s}{\beta c} \right)^2 \frac{z}{\eta} \frac{\partial f_1}{\partial \delta} \right) \right) \right. \\
&\quad \times e^{-i\Omega s/(\beta c)} \\
&= \left(-f_1 \frac{Q_y}{R} \frac{\partial g_1}{\partial \theta} - g_1 \frac{\omega_s}{\beta c} \frac{\partial f_1}{\partial \phi} \right) e^{-i\Omega s/(\beta c)}
\end{aligned}
$$

Vlasov eq. – 2-dimensional system

Step #1: evaluate first Poisson bracket

The Poisson bracket becomes:

$$[H_0, \psi_1] = \left(f_1 \left(\left(\frac{Q_y}{R} \right)^2 y \frac{\partial g_1}{\partial p_y} - p_y \frac{\partial g_1}{\partial y} \right) + \left(g_1 \left(\eta\, \delta \frac{\partial f_1}{\partial z} - \left(\frac{\omega_s}{\beta c} \right)^2 \frac{z}{\eta} \frac{\partial f_1}{\partial \delta} \right) \right) \right)$$

$$\times\, e^{-i\Omega s/(\beta c)}$$

$$= \left(-f_1 \frac{Q_y}{R} \frac{\partial g_1}{\partial \theta} - g_1 \frac{\omega_s}{\beta c} \frac{\partial f_1}{\partial \phi} \right) e^{-i\Omega s/(\beta c)}$$

Why did we not perform the transformation on the Hamiltonian directly but instead only transform after the evaluation of the Poisson brackets?

Vlasov eq. – 2-dimensional system

Step #2: evaluate second Poisson bracket

The Poisson bracket becomes:

$$[H_0,\psi_1] = \left(f_1 \left(\left(\frac{Q_y}{R}\right)^2 y \frac{\partial g_1}{\partial p_y} - p_y \frac{\partial g_1}{\partial y} \right) + \left(g_1 \left(\eta\,\delta \frac{\partial f_1}{\partial z} - \left(\frac{\omega_s}{\beta c}\right)^2 \frac{z}{\eta} \frac{\partial f_1}{\partial \delta} \right)\right)\right.$$
$$\times\, e^{-i\Omega s/(\beta c)}$$
$$= \left(-f_1 \frac{Q_y}{R} \frac{\partial g_1}{\partial \theta} - g_1 \frac{\omega_s}{\beta c} \frac{\partial f_1}{\partial \phi} \right) e^{-i\Omega s/(\beta c)}$$

Because $[H_0 + H_1(\psi_0), \psi_0] = 0$ it follows that $g_0(y,p_y) = g_0(J_y)$ and $f_0(z,\delta) = f_0(r)$. Then, the second Poisson bracket evaluates to

$$[H_1,\psi_0] = f_0 \frac{\partial H_1}{\partial y} \frac{\partial g_0}{\partial p_y} + g_0 \frac{\partial H_1}{\partial z} \frac{\partial f_0}{\partial \delta}$$
$$= \frac{e^2}{\beta^2 E C} \left(f_0 \sqrt{\frac{2 J_y R}{Q_y}} \sin\theta\, g_0' \frac{y^n}{n!} V(z) + g_0 \frac{\eta \beta c}{\omega_s} \sin\phi\, f_0' \int dy\, \frac{y^n}{n!} \frac{\partial}{\partial z} V(z) \right)$$

Vlasov eq. – 2-dimensional system

Step #2: evaluate second Poisson bracket

The Poisson bracket becomes:

$$[H_0, \psi_1] = \left(f_1 \left(\left(\frac{Q_y}{R} \right)^2 y \frac{\partial g_1}{\partial p_y} - p_y \frac{\partial g_1}{\partial y} \right) + \left(g_1 \left(\eta \, \delta \frac{\partial f_1}{\partial z} - \left(\frac{\omega_s}{\beta c} \right)^2 \frac{z}{\eta} \frac{\partial f_1}{\partial \delta} \right) \right) \right.$$
$$\times \, e^{-i\Omega s/(\beta c)}$$
$$= \left(-f_1 \frac{Q_y}{R} \frac{\partial g_1}{\partial \theta} - g_1 \frac{\omega_s}{\beta c} \frac{\partial f_1}{\partial \phi} \right) e^{-i\Omega s/(\beta c)}$$

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$$[H_1, \psi_0] = f_0 \frac{\partial H_1}{\partial y} \frac{\partial g_0}{\partial p_y} + g_0 \frac{\partial H_1}{\partial z} \frac{\partial f_0}{\partial \delta}$$

negligible in practice (s. Ex 6.18)

$$= \frac{e^2}{\beta^2 E C} \left(f_0 \sqrt{\frac{2 J_y R}{Q_y}} \sin\theta \, g_0' \, \frac{y^n}{n!} V(z) + g_0 \frac{\eta \beta c}{\omega_s} \sin\phi \, f_0' \int dy \, \frac{y^n}{n!} \frac{\partial}{\partial z} V(z) \right)$$

Vlasov eq. – 2-dimensional system

Step #2: evaluate second Poisson bracket

We then write the Vlasov equation with the evaluated Poisson brackets as

$$
\begin{aligned}
-i\frac{\Omega}{\beta c} f_1 g_1 e^{-i\Omega s/(\beta c)} &= \left(-f_1 \frac{Q_y}{R}\frac{\partial g_1}{\partial \theta} - g_1 \frac{\omega_s}{\beta c}\frac{\partial f_1}{\partial \phi}\right) e^{-i\Omega s/(\beta c)} \\
&+ \frac{e^2}{\beta^2 E C} f_0 \sqrt{\frac{2 J_y R}{Q_y}} \sin\theta \, g_0' \, \frac{y^n}{n!} \\
&\times V(z)
\end{aligned}
$$

Vlasov eq. – 2-dimensional system

Step #2: evaluate second Poisson bracket

We then write the Vlasov equation with the evaluated Poisson brackets as

$$-i\frac{\Omega}{\beta c}f_1 g_1 e^{-i\Omega s/(\beta c)} = \left(-f_1\frac{Q_y}{R}\frac{\partial g_1}{\partial\theta} - g_1\frac{\omega_s}{\beta c}\frac{\partial f_1}{\partial\phi}\right)e^{-i\Omega s/(\beta c)}$$
$$+\frac{e^2}{\beta^2 EC}f_0\sqrt{\frac{2J_yR}{Q_y}}\sin\theta\, g_0'\,\frac{y^n}{n!}$$
$$\times\int dz' \sum_{k=-\infty}^{\infty}\rho^{(m)}(z')e^{-i\Omega(s/(\beta c)-kT_0)}W_{mn}(z-z'-kcT_0)$$

- This looks a lot more complicated than its longitudinal countepart. What will save us, here, is that we restrict ourselves to purely dipolar transverse motion (i.e. m=1, n=0)
- This will enable us to factor out the transverse dimension from the Vlasov equation and reduce the problem to nearly the same one that has already been solved for the longitudinal plane.

Vlasov eq. – 2-dimensional system

Step #3: factorize transverse dimension

Fixing m=1 and n=0, we rewrite the Vlasov equation as

$$
\begin{aligned}
-i\frac{\Omega}{\beta c} f_1 g_1 e^{-i\Omega s/(\beta c)} = & \left(-f_1 \frac{Q_y}{R}\frac{\partial g_1}{\partial \theta} - g_1 \frac{\omega_s}{\beta c}\frac{\partial f_1}{\partial \phi}\right) e^{-i\Omega s/(\beta c)} \\
& + \frac{e^2}{\beta^2 E C} f_0 \sqrt{\frac{2 J_y R}{Q_y}} \sin\theta \, g_0' \\
& \times \int dz' \sum_{k=-\infty}^{\infty} \rho^{(1)}(z') e^{-i\Omega(s/(\beta c) - k T_0)} W_1(z - z' - k c T_0)
\end{aligned}
$$

- Let's pause a minute to try and understand some of the structure of the equation above.

Vlasov eq. – 2-dimensional system

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Fixing m=1 and n=0, we rewrite the Vlasov equation as

$$-i\frac{\Omega}{\beta c}\boxed{f_1}\boxed{g_1}e^{-i\Omega s/(\beta c)} = \left(-\boxed{f_1}\boxed{\frac{Q_y}{R}\frac{\partial g_1}{\partial\theta}} - \boxed{g_1}\boxed{\frac{\omega_s}{\beta c}\frac{\partial f_1}{\partial\phi}}\right)e^{-i\Omega s/(\beta c)}$$

$$+\frac{e^2}{\beta^2 EC}\boxed{f_0}\boxed{\sqrt{\frac{2J_yR}{Q_y}}\sin\theta\, g_0'}$$

$$\times\boxed{\int dz' \sum_{k=-\infty}^{\infty}\boxed{\rho^{(1)}}(z')e^{-i\Omega(s/(\beta c)-kT_0)}W_1(z-z'-kcT_0)}$$

Longitudinal structure | **Transverse structure**

Vlasov eq. – 2-dimensional system

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$$+\frac{e^2}{\beta^2 EC}\boxed{f_0}\boxed{\sqrt{\frac{2J_y R}{Q_y}}\sin\theta\, g_0'}$$
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Separable –
Fourier decompositions
in angles

Longitudinal structure | **Transverse structure**

Vlasov eq. – 2-dimensional system

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&+ \frac{e^2}{\beta^2 EC} f_0 \sqrt{\frac{2J_y R}{Q_y}}\sin\theta\, g_0' \\
&\times \int dz' \sum_{k=-\infty}^{\infty} \rho^{(1)}(z')e^{-i\Omega(s/(\beta c)-kT_0)}W_1(z-z'-kcT_0)
\end{aligned}
$$

Simple transverse structure (dipolar)

Longitudinal structure | **Transverse structure**

Vlasov eq. – 2-dimensional system

Step #3: factorize transverse dimension

Fixing m=1 and n=0, we rewrite the Vlasov equation as

$$-i\frac{\Omega}{\beta c}\boxed{f_1}\boxed{g_1}e^{-i\Omega s/(\beta c)} = \left(-\boxed{f_1}\frac{Q_y}{R}\frac{\partial g_1}{\partial \theta} - \boxed{g_1}\frac{\omega_s}{\beta c}\frac{\partial f_1}{\partial \phi}\right) e^{-i\Omega s/(\beta c)}$$

$$+\frac{e^2}{\beta^2 EC} f_0 \sqrt{\frac{2J_y R}{Q_y}} \sin\theta\, g_0'$$

$$\times \int dz' \sum_{k=-\infty}^{\infty} \rho^{(1)}(z') e^{-i\Omega(s/(\beta c)-kT_0)} W_1(z-z'-kcT_0)$$

Simple transverse structure (dipolar)

Much more complicated longitudinal structure

Longitudinal structure | **Transverse structure**

Vlasov eq. – 2-dimensional system

Step #3: factorize transverse dimension

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$$+ \frac{e^2}{\beta^2 EC} f_0 \sqrt{\frac{2J_y R}{Q_y}} \sin\theta \, g_0'$$
$$\times \int dz' \sum_{k=-\infty}^{\infty} \rho^{(1)}(z') e^{-i\Omega(s/(\beta c) - kT_0)} W_1(z - z' - kcT_0)$$

- Before moving on, let's allow for another minor subtlety

Vlasov eq. – 2-dimensional system

Step #3: factorize transverse dimension

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$$-i\frac{\Omega}{\beta c}f_1 g_1 e^{-i\Omega s/(\beta c)} = \left(-f_1\frac{Q_y}{R}\frac{\partial g_1}{\partial\theta} - g_1\frac{\omega_s}{\beta c}\frac{\partial f_1}{\partial\phi}\right)e^{-i\Omega s/(\beta c)}$$

$$+\frac{e^2}{\beta^2 EC}f_0\sqrt{\frac{2J_yR}{Q_y}}\sin\theta\, g_0'$$

$$\times\int dz'\sum_{k=-\infty}^{\infty}\rho^{(1)}(z')e^{-i\Omega(s/(\beta c)-kT_0)}W_1(z-z'-kcT_0)$$

we include chromatic effects (another coupling from longitudinal to transverse):

$Q_y = Q_{y0} + Q_y'\delta$

- Before moving on, let's allow for another minor subtlety

Vlasov eq. – 2-dimensional system

Step #3: factorize transverse dimension

We now introduce the following decompositions

$$g_1(J_y, \phi) = \sum_k g_1^{(k)}(J_y) e^{ik\theta}$$

Comparing with the RHS of the Vlasov equation, it can be shown that

$$g_1^{(k)}(J_y) = 0\,, \quad \forall k \setminus \{-1, 1\}$$

The $k = -1$ solution can be neglected assuming $|\Omega - \omega_y| \ll |\Omega + \omega_y|$. Hence,

$$\boxed{g_1(J_y, \theta) = g(J_y)\, e^{i\theta}\,.}$$

And

$$\boxed{f_1(r, \phi) = e^{iQ'_y z/(\eta R)} \sum_l a_l R_l(r) e^{il\phi}}$$

Vlasov eq. – 2-dimensional system

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$$\boxed{g_1(J_y, \theta) = g(J_y)\, e^{i\theta}\,.}$$

And

$$\boxed{f_1(r, \phi) = e^{iQ'_y z/(\eta R)} \sum_l a_l R_l(r) e^{il\phi}}$$

Very similar to what we had earlier, now, with a coefficient that will pull the chromatic dependence in the eigenvalues up into the phase of the eigenvectors – this will have important consequences and manifest into the slow headtail instabilities

Vlasov eq. – 2-dimensional system

Step #3: factorize transverse dimension

Inserting the decompositions above we arrive at the Vlasov equation

$$\sum_{l''} a_{l''} R_{l''}(r) e^{il''\phi} \frac{g(J_y)(\Omega - \omega_{y0} - l''\omega_s)}{g_0'(J_y)\sqrt{\frac{2J_y R}{Q_y}}} e^{-i\Omega s/(\beta c)} = \frac{e^2 c}{2\beta E C} f_0\, e^{-iQ_y' z/(\eta R)}$$

$$\times \int dz' \sum_{k=-\infty}^{\infty} \rho^{(1)}(z') e^{-i\Omega(s/(\beta c) - kT_0)} W_1(z - z' - kcT_0)$$

Vlasov eq. – 2-dimensional system

Step #3: factorize transverse dimension

Inserting the decompositions above we arrive at the Vlasov equation

$$\sum_{l''} a_{l''} R_{l''}(r) e^{il''\phi} \underbrace{\frac{g(J_y)(\Omega - \omega_{y0} - l''\omega_s)}{g_0'(J_y)\sqrt{\frac{2J_y R}{Q_y}}}} e^{-i\Omega s/(\beta c)} = \frac{e^2 c}{2\beta E C} f_0 \, e^{-iQ_y' z/(\eta R)}$$

constant in J_y, therefore $\dfrac{g(J_y)}{g_0'(J_y)\sqrt{\frac{2J_y R}{Q_y}}} = D$

$$\times \int dz' \sum_{k=-\infty}^{\infty} \rho^{(1)}(z') e^{-i\Omega(s/(\beta c) - kT_0)} W_1(z - z' - kcT_0)$$

Vlasov eq. – 2-dimensional system

Step #3: factorize transverse dimension

Inserting the decompositions above we arrive at the Vlasov equation

$$\sum_{l''} a_{l''} R_{l''}(r) e^{il''\phi} \underbrace{\frac{g(J_y)(\Omega - \omega_{y0} - l''\omega_s)}{g_0'(J_y)\sqrt{\frac{2J_y R}{Q_y}}}} e^{-i\Omega s/(\beta c)} = \frac{e^2 c}{2\beta E C} f_0\, e^{-iQ_y' z/(\eta R)}$$

constant in J_y, therefore $\dfrac{g(J_y)}{g_0'(J_y)\sqrt{\frac{2J_y R}{Q_y}}} = D$

$$\times \int dz' \sum_{k=-\infty}^{\infty} \underbrace{\rho^{(1)}(z')} e^{-i\Omega(s/(\beta c) - kT_0)} W_1(z - z' - kcT_0)$$

$$\rho^{(1)}(z) = \frac{\int dJ_y\, d\theta\, g(J_y) e^{i\theta}\, y}{\int dJ_y\, d\theta\, g_0(J_y)} \rho(z) = D\frac{R}{Q_{y0}} \frac{\int dJ_y\, g_0'(J_y)\, J_y}{\int dJ_y\, g_0(J_y)} \rho(z) = D\frac{R}{Q_{y0}} \rho(z)$$

Vlasov eq. – 2-dimensional system

Step #3: factorize transverse dimension

We get to

$$\sum_{l''} a_{l''} R_{l''}(r) e^{il''\phi} \left(\Omega - \omega_{y0} - l''\omega_s\right) e^{-i\Omega s/(\beta c)} = \frac{e^2}{2EC}\frac{c^2}{\omega_{y0}} f_0\, e^{-iQ'_y z/(\eta R)}$$
$$\times \int dz' \sum_{k=-\infty}^{\infty} \rho(z') e^{-i\Omega(s/(\beta c) - kT_0)} W_1(z - z' - kcT_0)$$

We have eliminated all dependencies on the transverse distribution functions and have arrived at the equivalent problem that we had already encountered during the longitudinal studies! Note that this was possible due to some certain assumptions we made, such as restricting our study to purely dipolar wakefield problems.

Vlasov eq. – 2-dimensional system

Step #3: factorize transverse dimension

We get to

$$\sum_{l''} a_{l''} R_{l''}(r) e^{il''\phi} \left(\Omega - \omega_{y0} - l''\omega_s\right) e^{-i\Omega s/(\beta c)} = \frac{e^2}{2EC}\frac{c^2}{\omega_{y0}} f_0\, e^{-iQ'_y z/(\eta R)}$$
$$\times \int dz' \sum_{k=-\infty}^{\infty} \rho(z') e^{-i\Omega(s/(\beta c) - kT_0)} W_1(z - z' - kcT_0)$$

We can at this stage follow the identical steps made for the longitudinal plane

- Move to frequency domain via the impedance involving the Poisson summation formula
- Multiply, integrate and use orthonormality of $e^{il\phi}$
- Using the inverse projection of the distribution function

Vlasov eq. – 2-dimensional system

Step #3: factorize transverse dimension

We get to

$$(\Omega - \omega_{y0} - l\omega_s) a_l R_l(r) = -i \frac{\pi e^2 \omega_s}{\eta \beta^2 E T_0^2 \omega_{y0}} f_0 \sum_{l'} \int r'\, dr'\, a_{l'} R_{l'}(r') i^{l-l'}$$

$$\times \sum_{p=-\infty}^{\infty} J_l\left(\frac{\omega' r}{\beta c} - \frac{Q'_{y0} r}{\eta R}\right) Z_1^{\perp}(\omega') J_{l'}\left(\frac{\omega' r'}{\beta c} - \frac{Q'_{y0} r'}{\eta R}\right)$$ [1]

Next, we perform the r-decomposition and introduce the weight function

- $R_l(r) = W(r) \sum_k b_{kl} u_{kl}(r)$
- $W(r) = \dfrac{\omega_s}{N \eta c} f_0(r)$
- We multiply and integrate by $\int r\, dr\, u_{kl}(r)$ and make use of the orthonormality conditions

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[1] $\omega' = \Omega - p\omega_0 \to p\omega_0 + \omega_{y0} + l\omega_s$

Vlasov eq. – 2-dimensional system

Step #4: formulate eigenvalue problem

We write the previous equation as

$$
\begin{aligned}
(\Omega - \omega_{y0} - l\omega_s)a_l b_{kl} &= -i\frac{\pi N e^2 c}{\beta^2 E T_0^2 \omega_{y0}} \sum_{k'} \sum_{l'} a_{l'} b_{k'l'}\, i^{l-l'} \\
&\times \sum_{p=-\infty}^{\infty} v_{kl}(\omega' - \omega_\xi)\, Z_1^\perp(\omega')\, v_{k'l'}(\omega' - \omega_\xi) \\
&= \sum_{k'} \sum_{l'} a_{l'}\, b_{k'l'}\, \mathcal{M}_{kk',ll'}
\end{aligned}
$$

with the interation matrix $\mathcal{M}_{kk',ll'}$ given as

$$
\mathcal{M}_{kk',ll'} = -i\frac{\pi N e^2 c}{\beta^2 E T_0^2 \omega_{y0}}\, i^{l-l'} \sum_{p=-\infty}^{\infty} v_{kl}(\omega' - \omega_\xi)\, Z_1^\perp(\omega')\, v_{k'l'}(\omega' - \omega_\xi)^1
$$

$\omega_\xi = Q'_y \omega_0 / \eta$

Vlasov eq. – 2-dimensional system

Step #4: formulate eigenvalue problem

We have finally arrived at a linear set of equations

$$(\Omega - \omega_{y0} - l\omega_s) a_l \, b_{kl} = \sum_{k'} \sum_{l'} a_{l'} \, b_{k'l'} \, \mathcal{M}_{kk',ll'} \, .$$

With

$$M_{kk',ll'} = l\omega_s \delta kk' \delta_{ll'} + \mathcal{M}_{kk',ll'} \, ,$$

this can be written as

$$\Big((\Omega - \omega_{y0}) \, \mathbf{1} - M \Big) \, v = 0 \, ,$$

a classical eigenvalue problem. We must, therefore, diagonalise the matrix M by solving the secular equation

$$\det \Big((\Omega - \omega_{y0}) \, \mathbf{1} - M \Big) = 0$$

to find the eigenvalues and the corresponding eigenvectors.

Some remarks on the EV problem

- The interaction matrix characterises the interaction of the basis functions with the impedance. The choice of basis functions is, in principle, arbitrary and will yield the same set of eigenvalues and eigenvectors while simply making the diagonalisation of the interaction matrix more or less tedious. The choices made here, with imposing the orthonormality conditions, were simply in order to obtain a symmetric form of the interaction matrix and to pre-solve the appearing integrals.

- The Vlasov solver DELPHI, for example, uses Laguerre polynomials for the expansion. Those correspond to the u_{kl} eigenvectors for Gaussian probability desity functions. Moreover, they can also be used for other distributions resulting, however, in more complicated integrals which can, nevertheless, be computed in a closed analytical form. It then diagonalises the interaction matrix resulting from the interaction of the Laguerre polynomial basis functions with the impedance.

Discussion of the EV problem

Let's take a look at the general structure of the eigenvalue problem and try to identify some particular cases. We express the eigenvalues probem as

$$E\, v = M\, v$$

with

$$E = \left(\frac{\Omega - \omega_{y0}}{\omega_s}\right)\mathbf{1}\,, \quad M_{kk',ll'} = l\,\delta_{kk',ll'} - i\,\frac{\pi N e^2 c}{\beta^2 E T_0^2 \omega_{y0}\omega_s}\, i^{l-l'}\,\langle kl|\, Z_1^{\perp}\,|k'l'\rangle$$

$$|kl\rangle = v_{kl}(\omega') = \int r\, dr\, W(r) u_{kl}(r) J_l\left(\frac{(\omega' - \omega_\xi)\, r}{\beta c}\right)$$

Discussion of the EV problem

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Head-Tail instability:
We neglect azimuthal mode coupling and stick to just one azimuthal mode, i.e. $l = l'$. Then, we can re-cast the constant term $l\omega_s\mathbf{1}$ into the eigenvalue, such that $E = (\Omega - \omega_{y0} - l\omega_s)\,\mathbf{1}$.
We now just need to diagonalise $\mathcal{M}$ with respect to the radial modes k:

$$\mathcal{M} = -i\,\frac{\pi N e^2 c}{\beta^2 E T_0^2 \omega_{y0}\omega_s}\, \langle kl|\, Z_1^{\perp}\, |k'l\rangle$$

Discussion of the EV problem

Let's take a look at the general structure of the eigenvalue problem and try to identify some particular cases. We express the eigenvalues probem as

$$E\, v = M\, v$$

with

$$E = \left(\frac{\Omega - \omega_{y0}}{\omega_s}\right)\mathbf{1}\,, \quad M_{kk',ll'} = \underbrace{l\,\delta_{kk',ll'} - i\,\frac{\pi N e^2 c}{\beta^2 E T_0^2 \omega_{y0}\omega_s}\, i^{l-l'}\,\langle kl|\, Z_1^{\perp}\, |k'l'\rangle}_{M}$$

Mode coupling instability:

We now consider azimuthal mode coupling and stick to just the dominant radial mode, i.e. $k = 0$. The term $l\omega_s\mathbf{1}$ is no longer constant but becomes a part of the interaction matrix.

We now need to diagonalise the full matrix M with respect to the azimuthal modes l:

$$M = l\,\delta_{ll'} - i\,\frac{\pi N e^2 c}{\beta^2 E T_0^2 \omega_{y0}\omega_s}\, i^{l-l'}\,\langle l|\, Z_1^{\perp}\, |l'\rangle$$

Slow headtail vs. fast headtail

Let's now take a look at the general form of the matrices to be diagonalised for the previous two cases. We had

$$\underbrace{\mathcal{M} = -i\frac{\pi N e^2 c}{\beta^2 E T_0^2 \omega_{y0}\omega_s}\left[m_{kk'}\right]}\ , \quad \underbrace{M = \begin{pmatrix} \ddots & & & & & & \\ & 2+I & R & I & R & I & \\ & R & 1+I & R & I & R & \\ & I & R & I & R & I & \\ & R & I & R & -1+I & R & \\ & I & R & R & R & -2+I & \\ & & & & & & \ddots \end{pmatrix}}$$

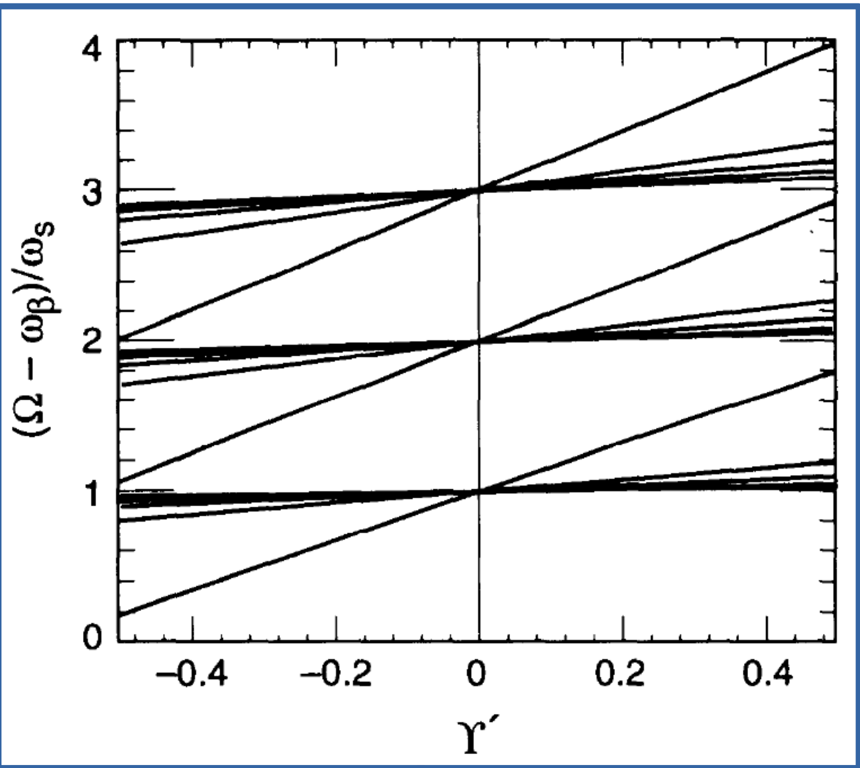
Slow headtail mode:
linear in intensity or shunt impedance $\rightarrow$ constant frequency shift for each radial mode.

Fast headtail mode:
complex interplay between real (R) and imaginary (I) parts of the impedance (all R and I are different). Off-diagonal elements are antisymmetric ($M_{-l,-l'} = -M_{l,l'}$). Non-linear in intensity or shunt impedance $\rightarrow$ azimuthal modes may couple!

Slow headtail vs. fast headtail

Let's now take a look at the general form ... or the previous two cases. We had

$$\mathcal{M} = -i\frac{\pi N e^2 c}{\beta^2 E T_0^2 \omega_{y0}\omega_s}\left[m_{kk'}\right] , \quad M = \begin{pmatrix} \ddots & & \\ & & I \\ & & \ddots \end{pmatrix}$$

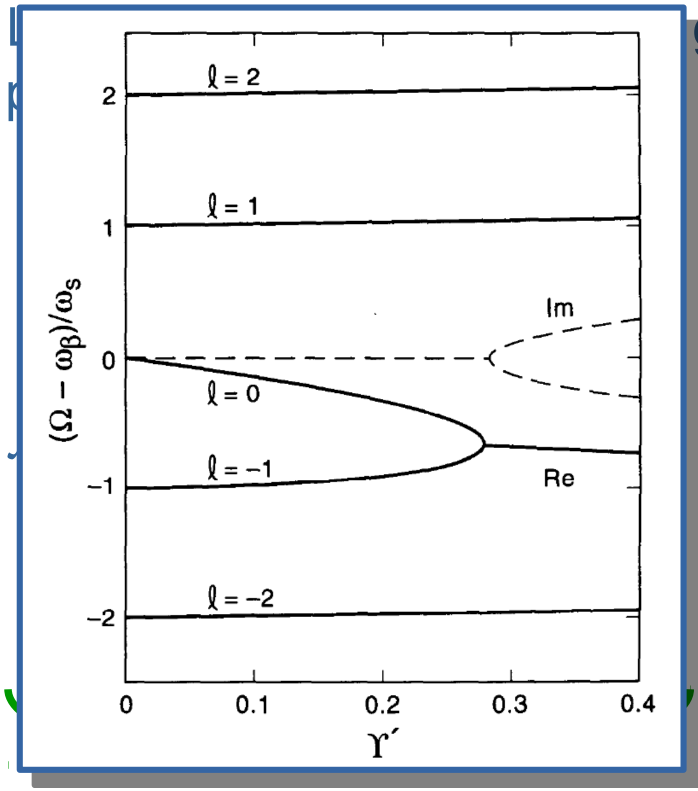


Mode frequencies vs intensity parameter of a parabolic beam in the presence of a resistive wall impedance

Slow headtail mode:
linear in intensity or shunt impedance $\rightarrow$ constant frequency shift for each radial mode.

Slow headtail vs. fast headtail

general form of the matrices to be diagonalised for the



$$M = \begin{pmatrix} \ddots & & & & & & \\ & 2+I & R & I & R & I & \\ & R & 1+I & R & I & R & \\ & I & R & I & R & I & \\ & R & I & R & -1+I & R & \\ & I & R & R & R & -2+I & \\ & & & & & & \ddots \end{pmatrix}$$

Mode frequencies vs. intensity parameter from an airbag beam. The solid line shows the tune shift. The dashed line indicates the rise time.

Fast headtail mode:
complex interplay between real (R) and imaginary (I) parts of the impedance (all R and I are different). Off-diagonal elements are antisymmetric $(M_{-l,-l'} = -M_{l,l'})$. Non-linear in intensity or shunt impedance $\rightarrow$ azimuthal modes may couple!

Slow headtail and effect. impedance

We will now look a little closer at the slow headtail modes. The interaction matrix was given as

$$\mathcal{M} = -i\frac{\pi N e^2 c}{\beta^2 E T_0^2 \omega_{y0}\omega_s} \langle kl| Z_1^{\perp} |k'l\rangle$$

The ket-vector $|kl\rangle$ is written explicitly as

$$|kl\rangle = v_{kl}(\omega') = \int r\,dr\,W(r)u_{kl}(r)J_l\left(\frac{(\omega' - \omega_\xi)\,r}{\beta c}\right) ,$$

where

$$\omega' = p\omega_0 + \omega_{y0} + l\omega_s$$

$$\omega_\xi = \frac{Q_y'\,\omega_0}{\eta}$$

$$\int r\,dr\,W(r)u_{kl}(r)u_{k'l'}(r') = \delta_{kk'}\delta_{ll'}$$

Slow headtail and effect. impedance

We will now look a little closer at the slow headtail modes. The interaction matrix was given as

$$\mathcal{M} = -i \frac{\pi N e^2 c}{\beta^2 E T_0^2 \omega_{y0} \omega_s} \langle kl | Z_1^\perp | k'l \rangle$$

Important feature – not present in the longitudinal case

The ket-vector $|kl\rangle$ is written explicitly as

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where

$$\omega' = p\omega_0 + \omega_{y0} + l\omega_s$$

$$\omega_\xi = \frac{Q_y' \, \omega_0}{\eta}$$

$$\int r\, dr\, W(r) u_{kl}(r) u_{k'l'}(r') = \delta_{kk'} \delta_{ll'}$$

Slow headtail and effect. impedance

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$$\mathcal{M} = -i \frac{\pi N e^2 c}{\beta^2 E T_0^2 \omega_{y0} \omega_s} \langle kl | Z_1^{\perp} | k' l \rangle$$

The ket-vector $|kl\rangle$ is written explicitly as

$$|kl\rangle = v_{kl}(\omega') = \int r\, dr\, W(r) u_{kl}(r) J_l \left(\frac{(\omega' - \omega_\xi)\, r}{\beta c} \right) .$$

Note that

$$\hat{\rho}_1^{(k,l)}(\omega) \sim i^{-l} v_{kl}(\omega)^{a} .$$

We define the effective impedance as

$$\left(Z_1^{\perp}\right)_{\mathrm{eff}} = \frac{\sum_{p=-\infty}^{\infty} Z_1^{\perp}(\omega') \, |\hat{\rho}_1^{(kl)}(\omega' - \omega_\xi)|^2}{\sum_{p=-\infty}^{\infty} |\hat{\rho}_1^{(kl)}(\omega' - \omega_\xi)|^2} .$$

[a] see Chao Eq. 6.103

Slow headtail and effect. impedance

We define the effective impedance as

$$\left(Z_1^{\perp}\right)_{\text{eff}} = \frac{\sum_{p=-\infty}^{\infty} Z_1^{\perp}(\omega') \, |\hat{\rho}_1^{(kl)}(\omega' - \omega_{\xi})|^2}{\sum_{p=-\infty}^{\infty} |\hat{\rho}_1^{(kl)}(\omega' - \omega_{\xi})|^2} \, .$$

With the notion of the effective impedance, assuming the interaction matrix has been diagonalised, the latter can be written as

$$\mathcal{M} = -i \frac{\pi N e^2 c}{\beta^2 E T_0^2 \omega_{y0} \omega_s} \left(Z_1^{\perp}\right)_{\text{eff}} \sum_{p=-\infty}^{\infty} |v_{kl}(\omega' - \omega_{\xi})|^2$$

Slow headtail and effect. impedance

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$$\left(Z_1^{\perp}\right)_{\mathrm{eff}} = \frac{\sum_{p=-\infty}^{\infty} Z_1^{\perp}(\omega')\,|\hat{\rho}_1^{(kl)}(\omega'-\omega_\xi)|^2}{\sum_{p=-\infty}^{\infty}|\hat{\rho}_1^{(kl)}(\omega'-\omega_\xi)|^2}.$$

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This is a fundamental result for the slow headtail modes:

- The complex tune shift is given by the overlap integral of the impedance and the mode power spectrum.
- Due to the chromatic shift, the mode frequency acquires an imaginary part. If $\mathrm{Re}\left(Z_1^{\perp}{}_{\mathrm{eff}}\right) < 0$ the beam becomes unstable.

Slow headtail and effect. impedance

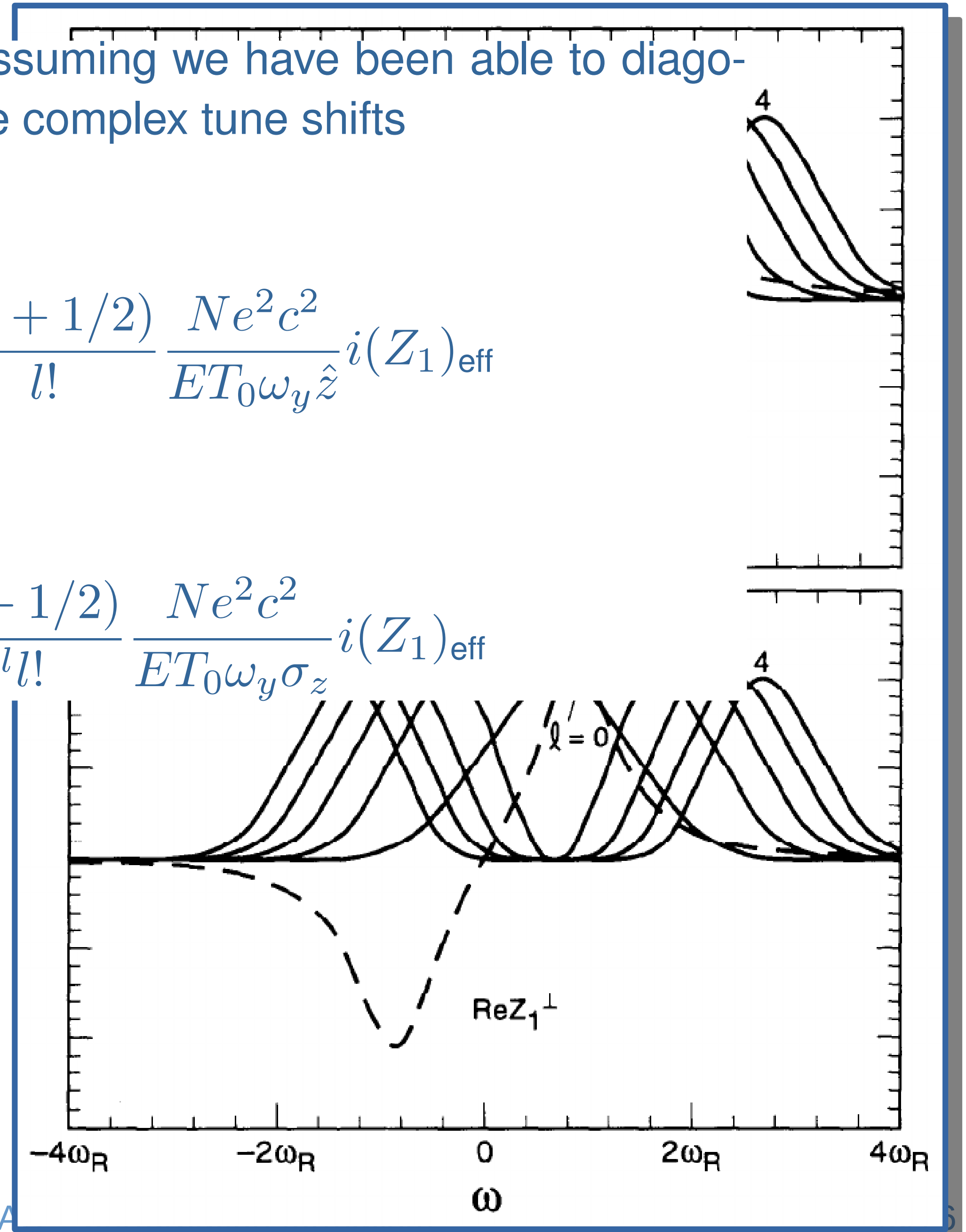
Considering the most prominent radial mode, assuming we have been able to diagonalise our matrix, we can actually compute some complex tune shifts

- Parabolic bunch

$$\Omega_l - \omega_y - l\omega_s = -\frac{1}{4\sqrt{\pi}}\frac{\Gamma(l+1/2)}{l!}\frac{Ne^2c^2}{ET_0\omega_y\hat{z}}i(Z_1)_{\text{eff}}$$

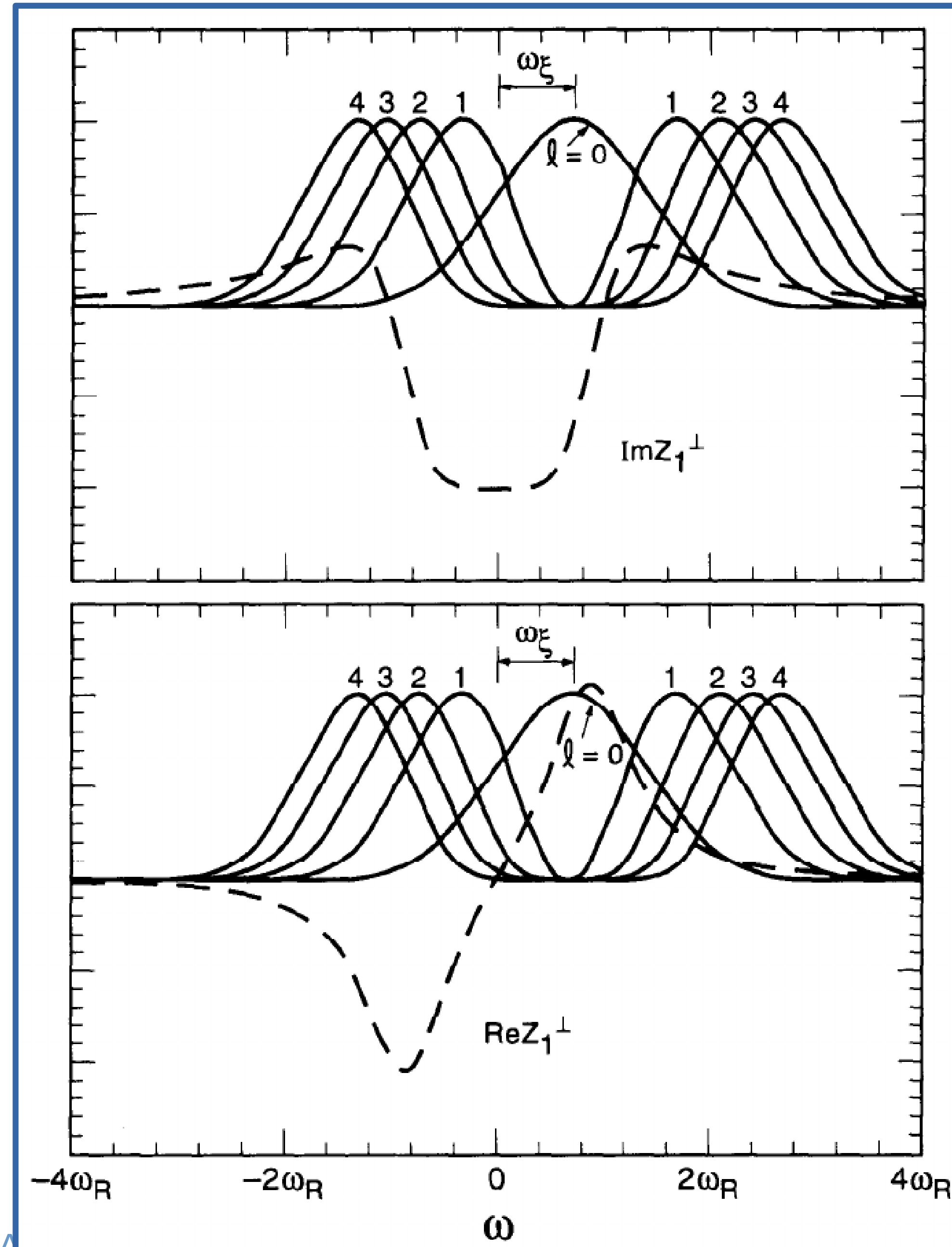
- Gaussian bunch

$$\Omega_l - \omega_y - l\omega_s = -\frac{1}{4\pi}\frac{\Gamma(l+1/2)}{2^l l!}\frac{Ne^2c^2}{ET_0\omega_y\sigma_z}i(Z_1)_{\text{eff}}$$



Slow headtail and effect. impedance

$$\left(Z_1^{\perp}\right)_{\mathrm{eff}} = \frac{\sum_{p=-\infty}^{\infty} Z_1^{\perp}(\omega') \, |\hat{\rho}_1^{(kl)}(\omega' - \omega_\xi)|^2}{\sum_{p=-\infty}^{\infty} |\hat{\rho}_1^{(kl)}(\omega' - \omega_\xi)|^2}$$



Look at the power spectrum and it's dependency on the slippage factor. What are the chromaticity settings you would preferably use in a machine operating below/above transition?

Another simple example

Let's assume an airbag distribution by assuming a weight function

$$W(r) = \frac{1}{2\pi z}\delta(r - \hat{z})\,,$$

so the radial part of the perturbation becomes

$$R_l(r) \propto \delta(r - \hat{z})\,.$$

There are infinite azimuthal modes, each mode resembling a particular stationary oscillation. Assuming for now $N = 0$, the zero intensity limit, it follows that the eingenvalue is readily obtained as

$$\Omega = \omega_{y0} + l\omega_s$$

and we can immediately write down the perturbation as

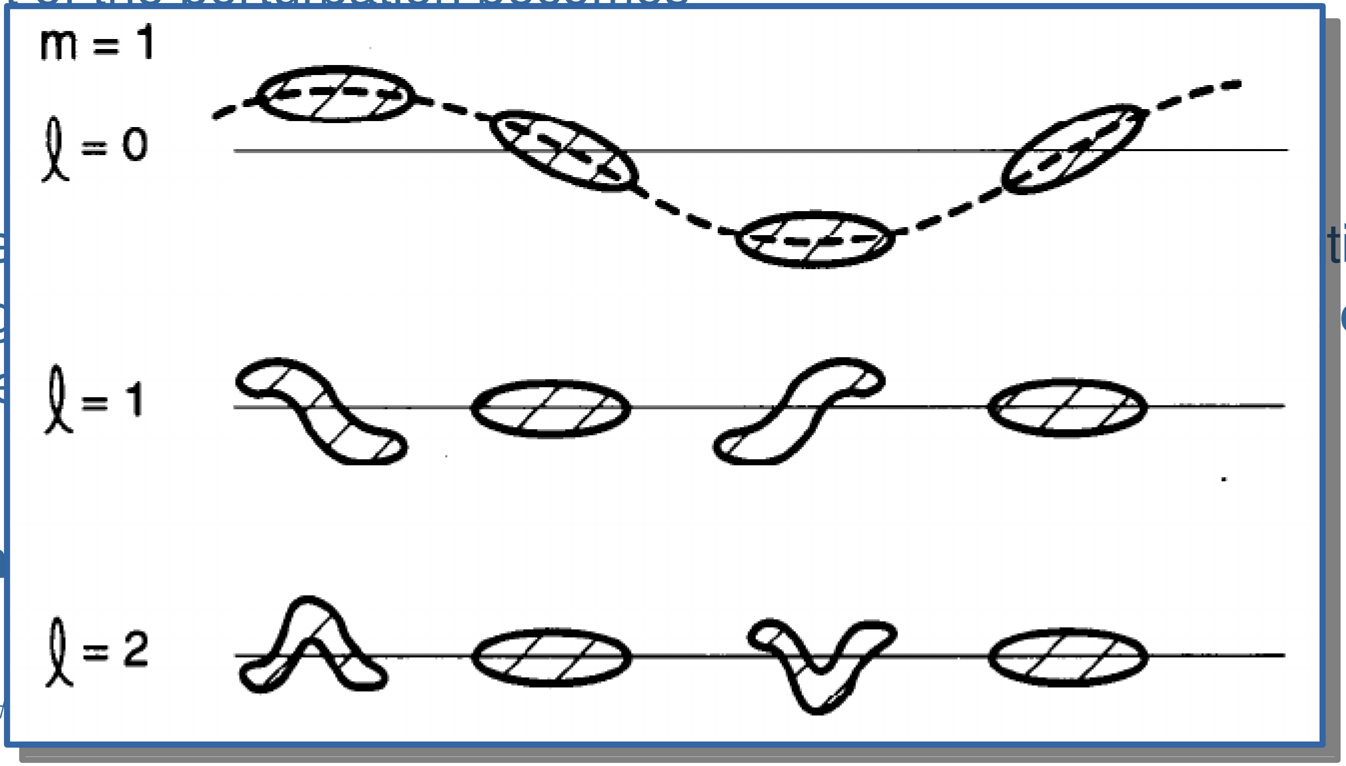
$$\psi_1 \propto g_0'(r)e^{i\theta}\,\delta(r - \hat{z})e^{il\phi}\,e^{iQ'z/(\eta R)}\,e^{-i(\omega_{y0}+l\omega_s)s/(\beta c)}$$

Another simple example

Let's assume an airbag distribution by assuming a weight function

$$W(r) = \frac{1}{2\pi z}\delta(r - \hat{z})\,,$$

so the radial part of the perturbation becomes



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Simulation results

We wight aswell solve the equations in time-domain using the original Hamiltonian

$$H = \frac{1}{2}p_y^2 + \left(\frac{Q_y}{R}\right)^2 y^2 - \frac{1}{2}\eta\,\delta^2 - \frac{1}{2\eta}\left(\frac{\omega_s}{\beta c}\right)^2 z^2$$
$$+ \frac{e^2}{\beta^2 EC}\, y \int dz' \sum_{k=-\infty}^{\infty} \rho^{(1)}(z') e^{-i\Omega(s/(\beta c)-kT_0)} W_{10}(z - z' - kcT_0)$$

Simulation results

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$$+ \frac{e^2}{\beta^2 EC}\, y \int dz' \sum_{k=-\infty}^{\infty} \rho^{(1)}(z') e^{-i\Omega(s/(\beta c) - kT_0)} W_{10}(z - z' - kcT_0)$$

dipolar kick (target)

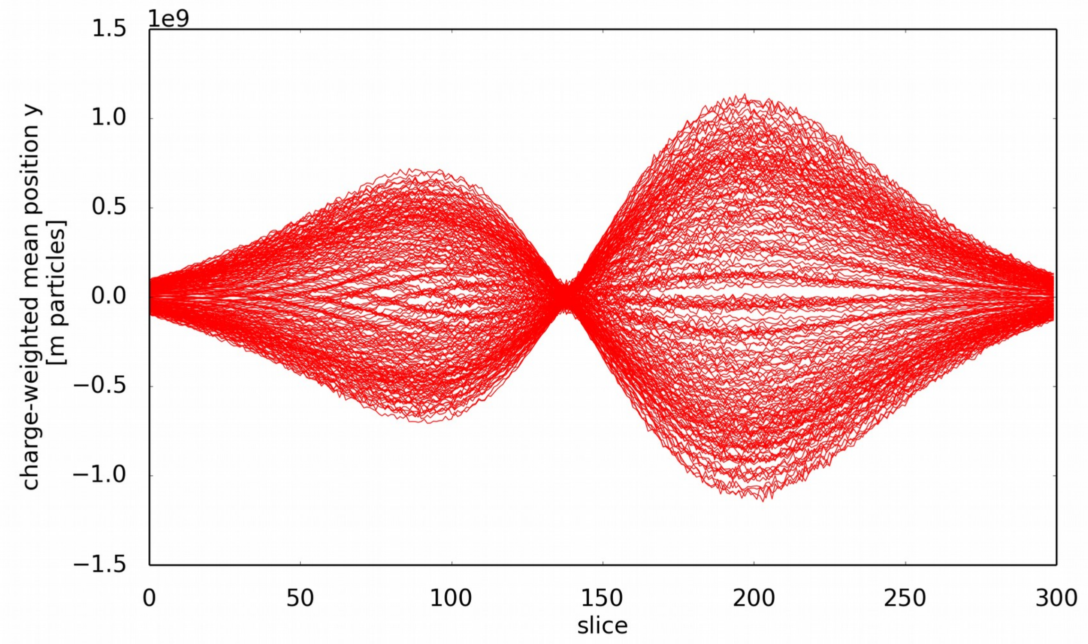
dipolar dependence (source)

What we will get is a position dependent orbit offset along the bunch which will turn out to be stationary (periodic in time). The resulting pictures are the manifestations of the different headtail modes which are obtained directly in the frequency domain calculations.

Simulation results – headtail modes

- Resistive wall wake – narrow band
- Consider negative real part of Z
- Exitation of single modes

Mode 1

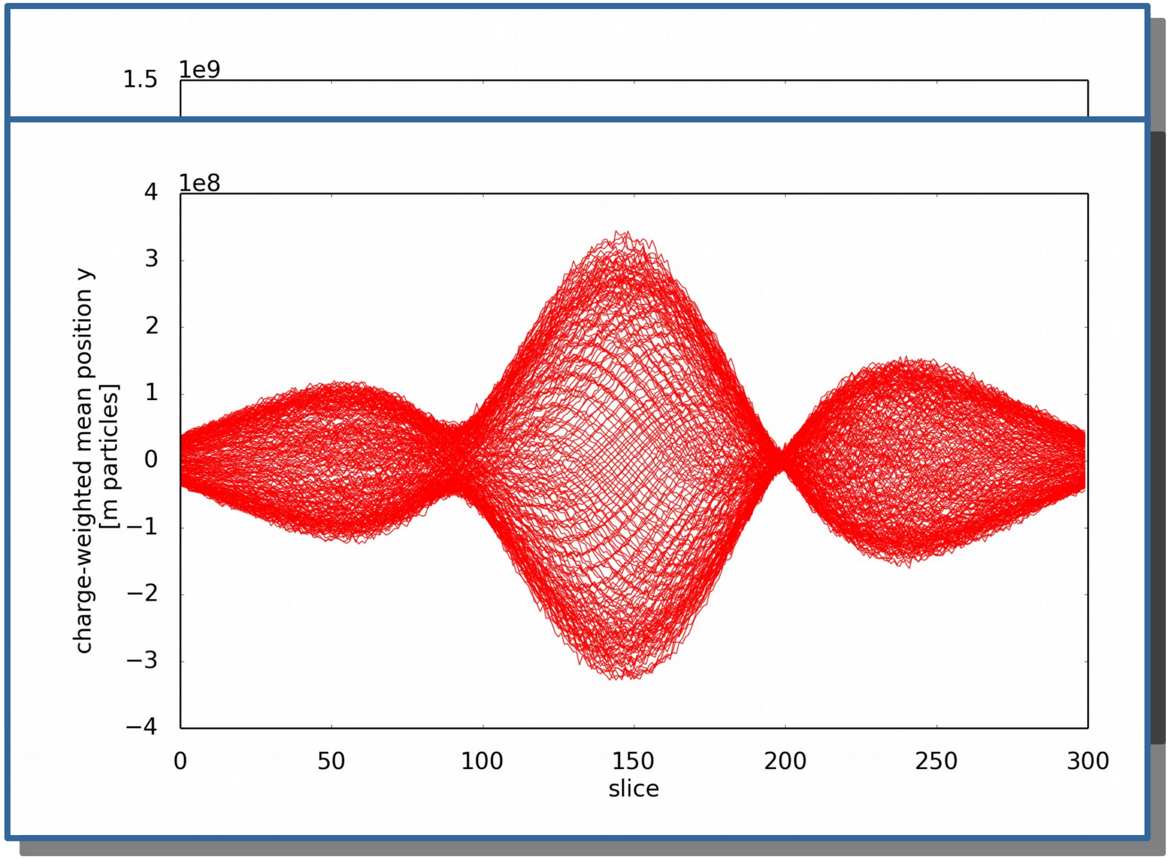


Simulation results – headtail modes

- Resistive wall wake – narrow band
- Consider negative real part of Z
- Exitation of single modes

Mode 1

Mode 2



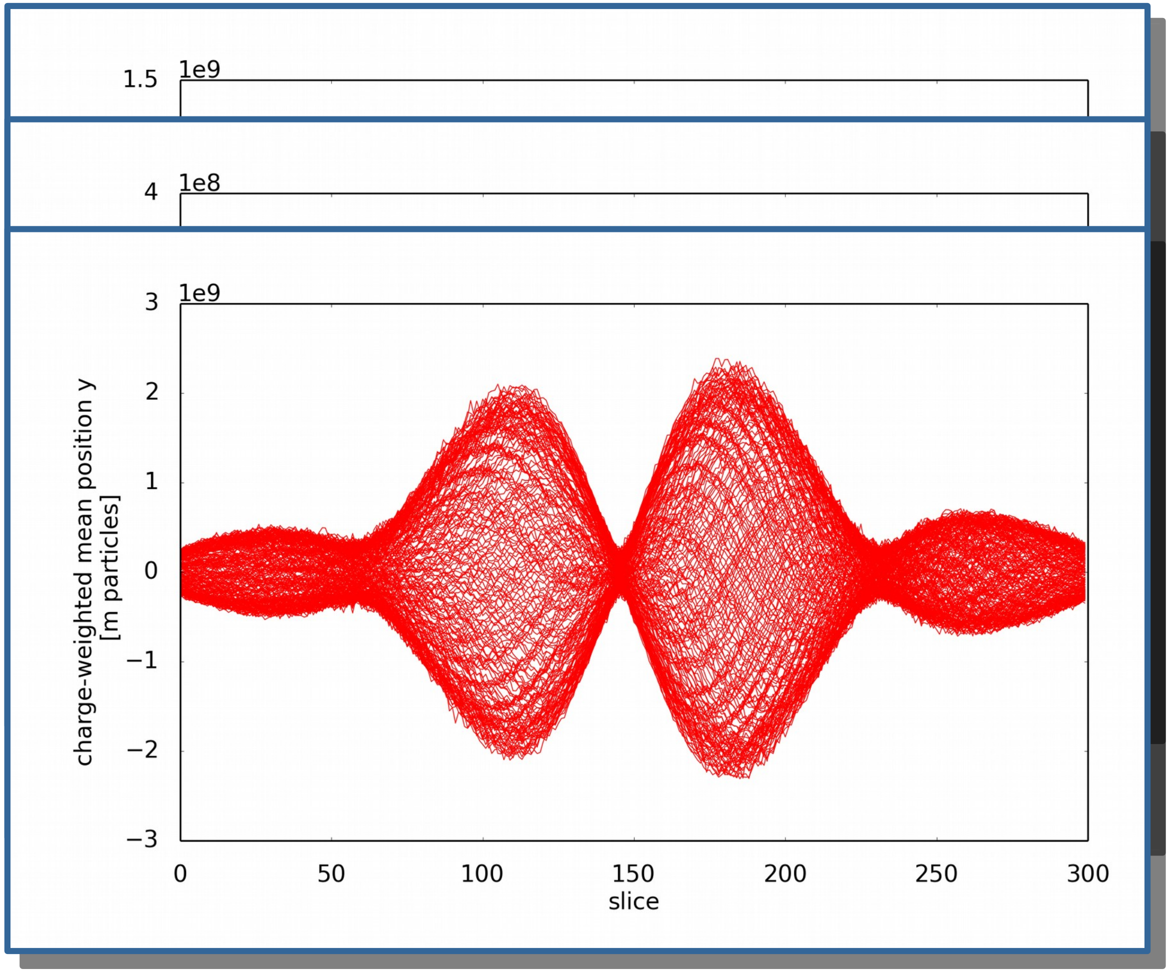
Simulation results – headtail modes

- Resistive wall wake – narrow band
- Consider negative real part of Z
- Exitation of single modes

Mode 1

Mode 2

Mode 3



Simulation results – headtail modes

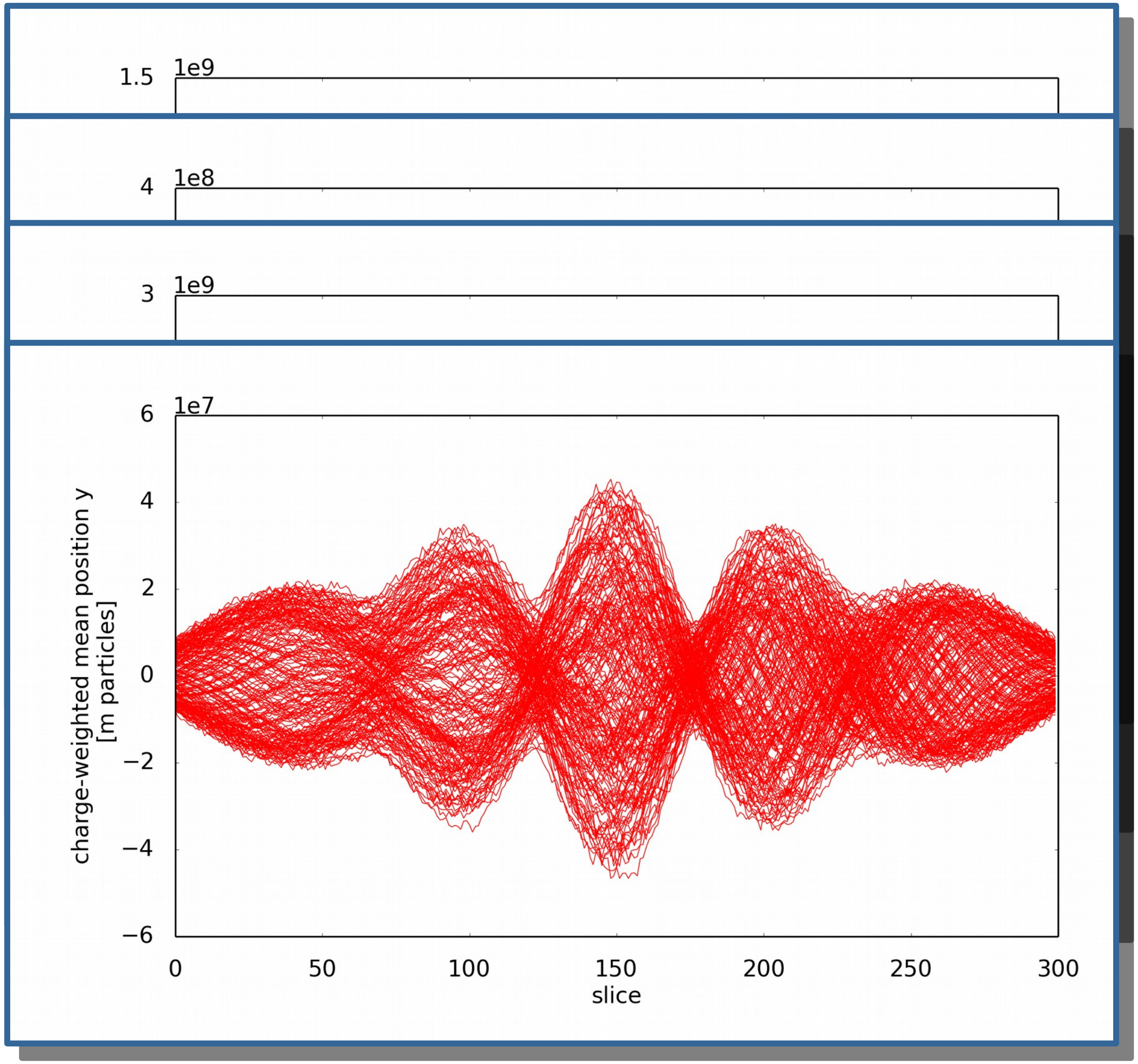
- Resistive wall wake – narrow band
- Consider negative real part of Z
- Exitation of single modes

Mode 1

Mode 2

Mode 3

Mode 4



Simulation results – headtail modes

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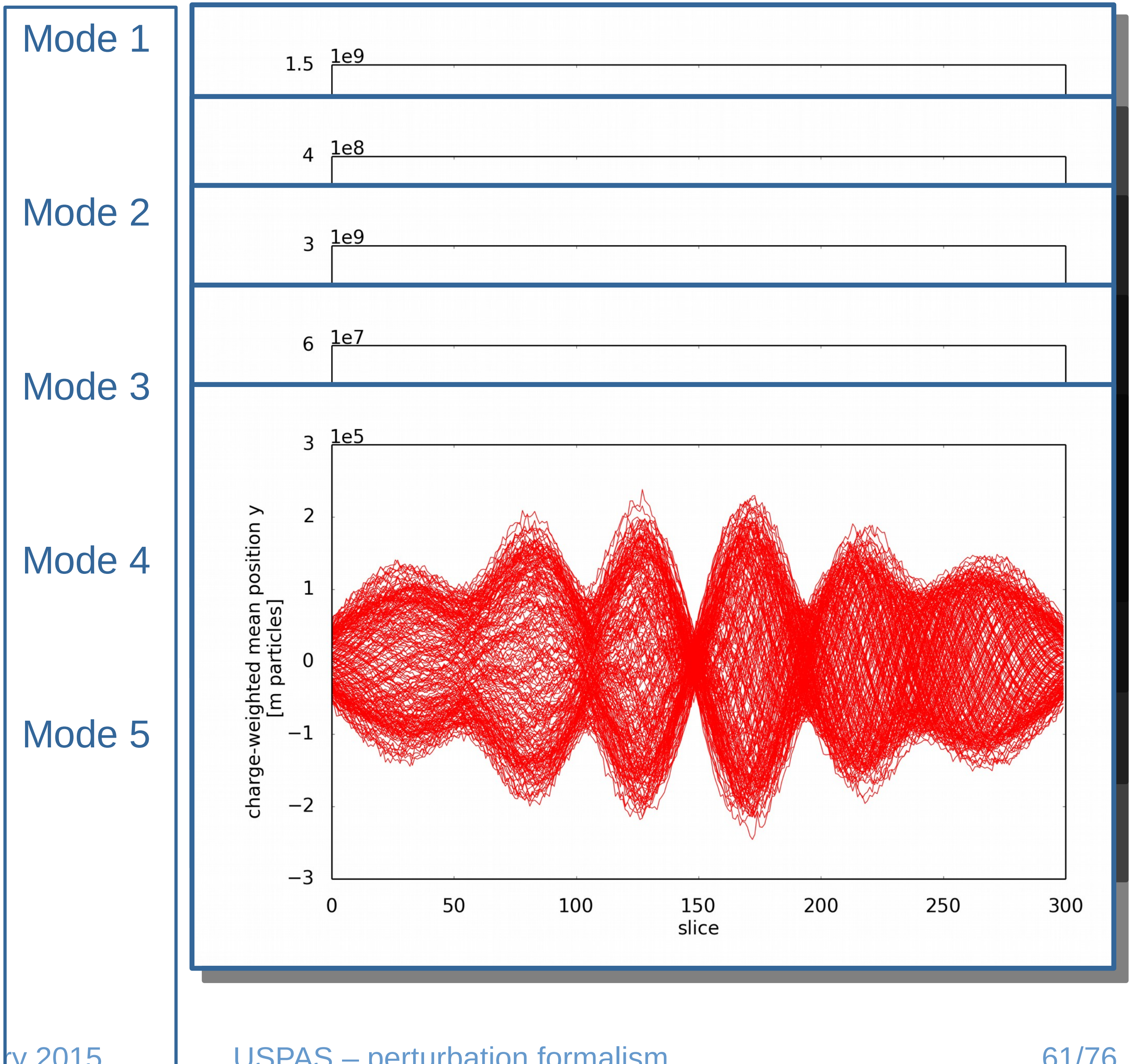


Simulation results – headtail modes

- Resistive wall wake – narrow band
- Consider negative real part of Z
- Exitation of single modes

Mode 1

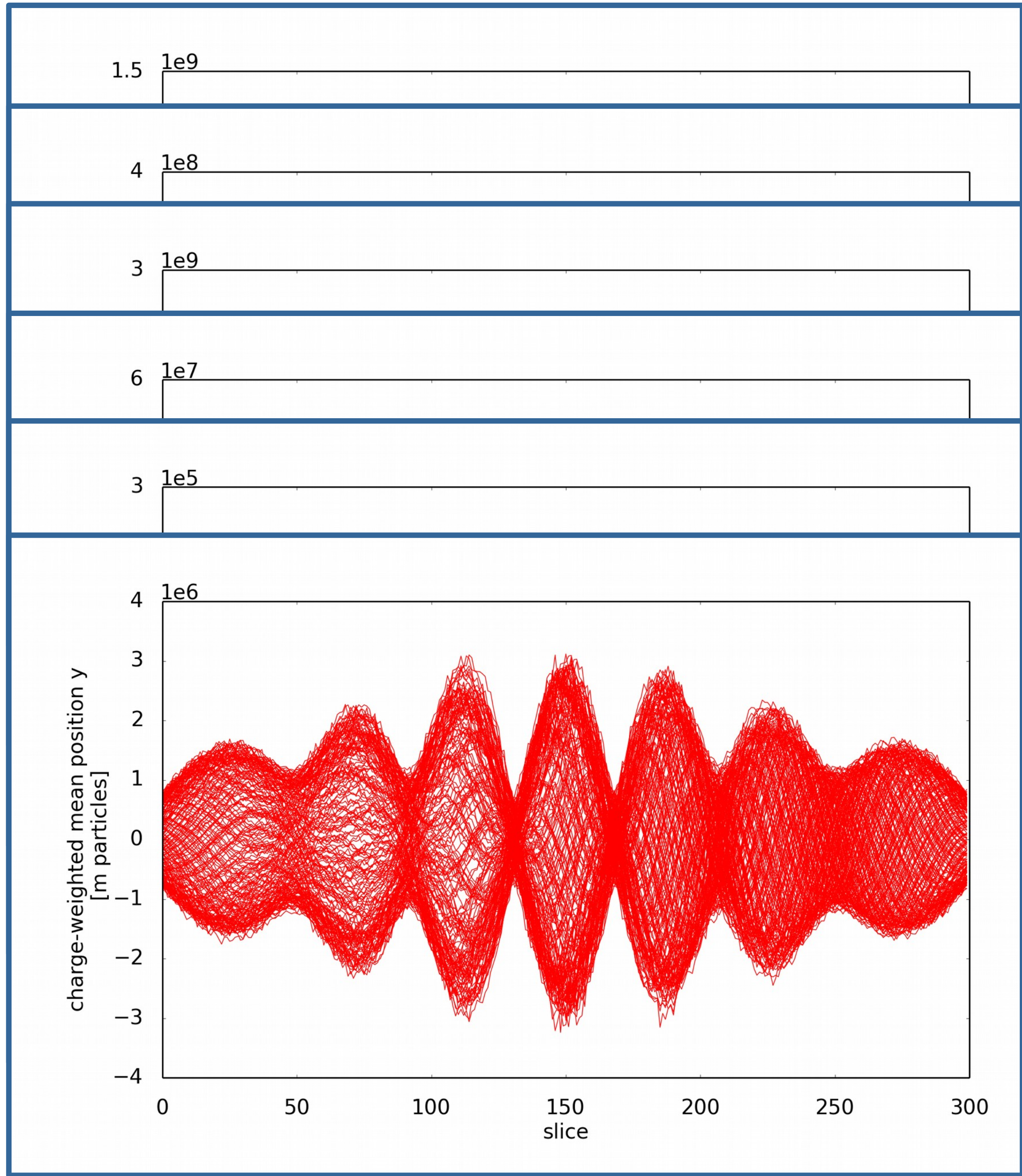
Mode 2

Mode 3

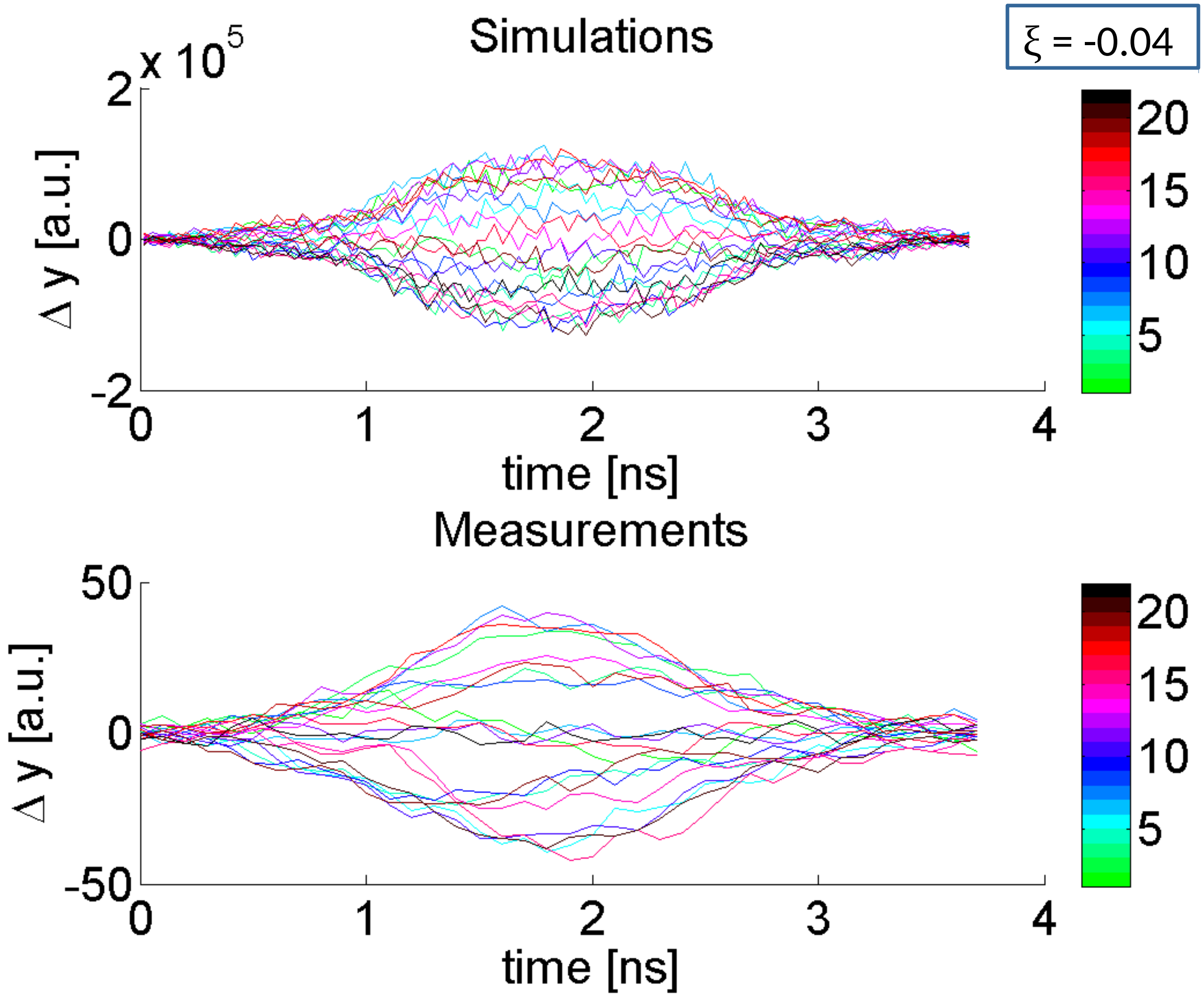
Mode 4

Mode 5

Mode 6



Impedance model: intra-bunch motion



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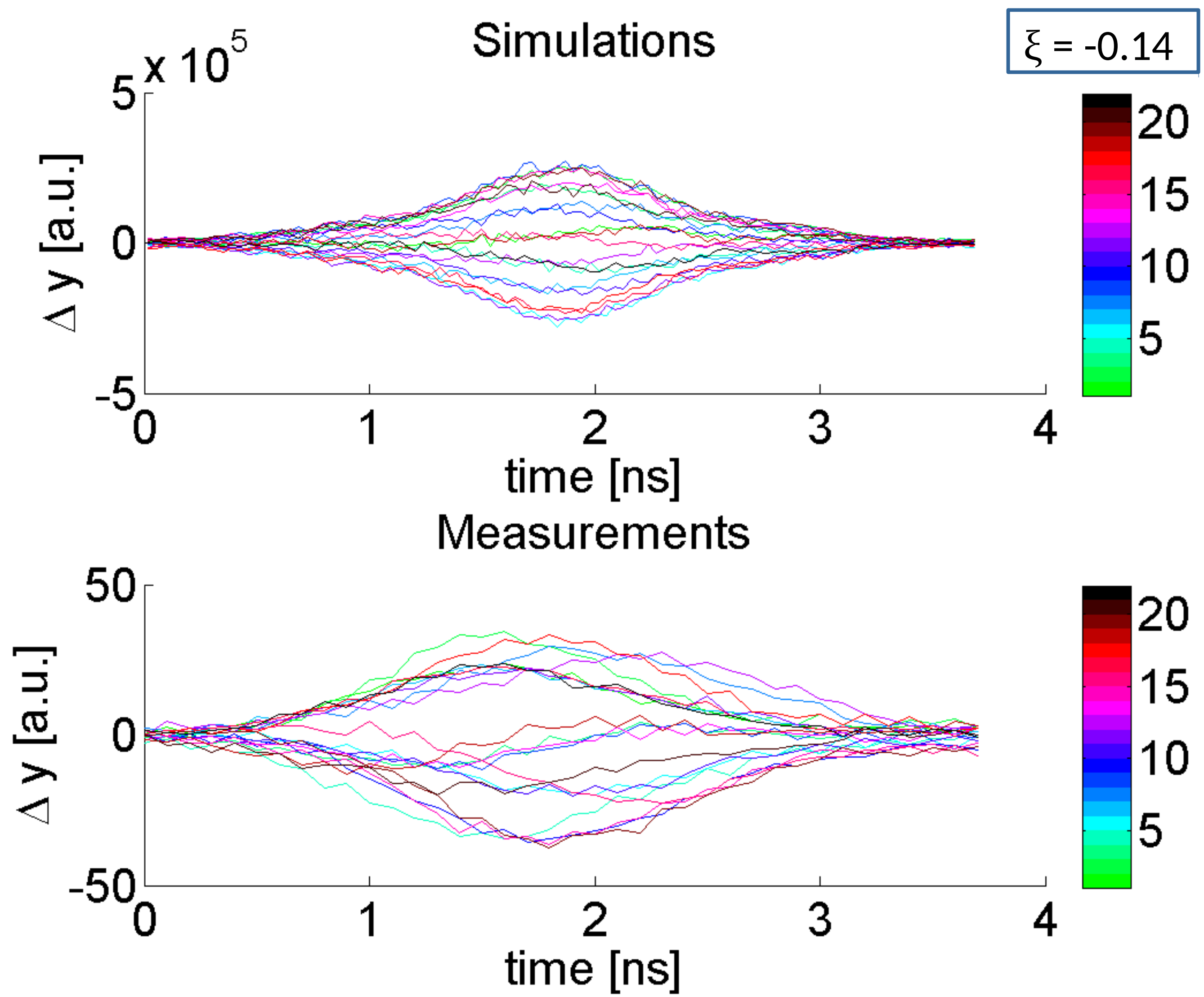
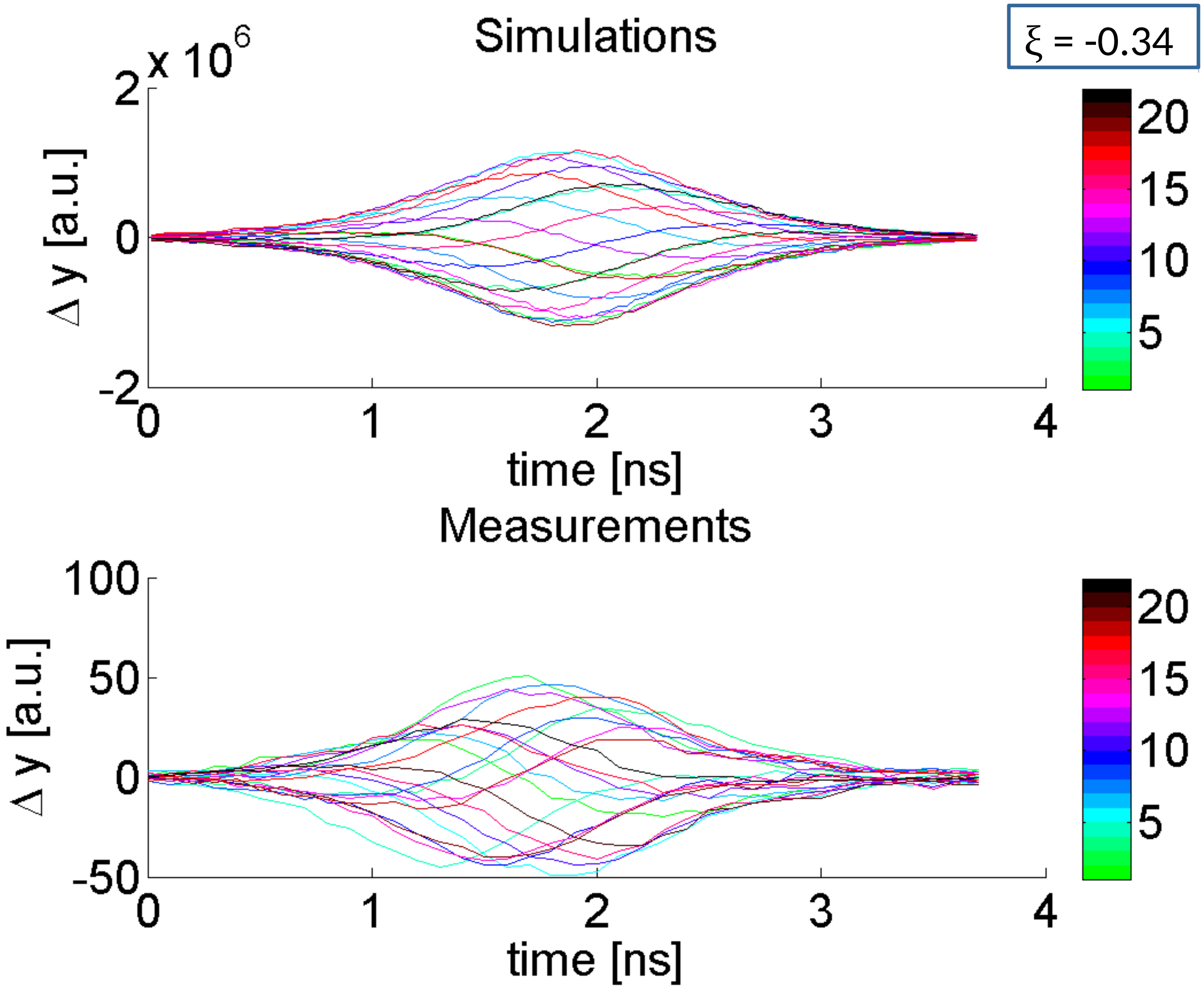
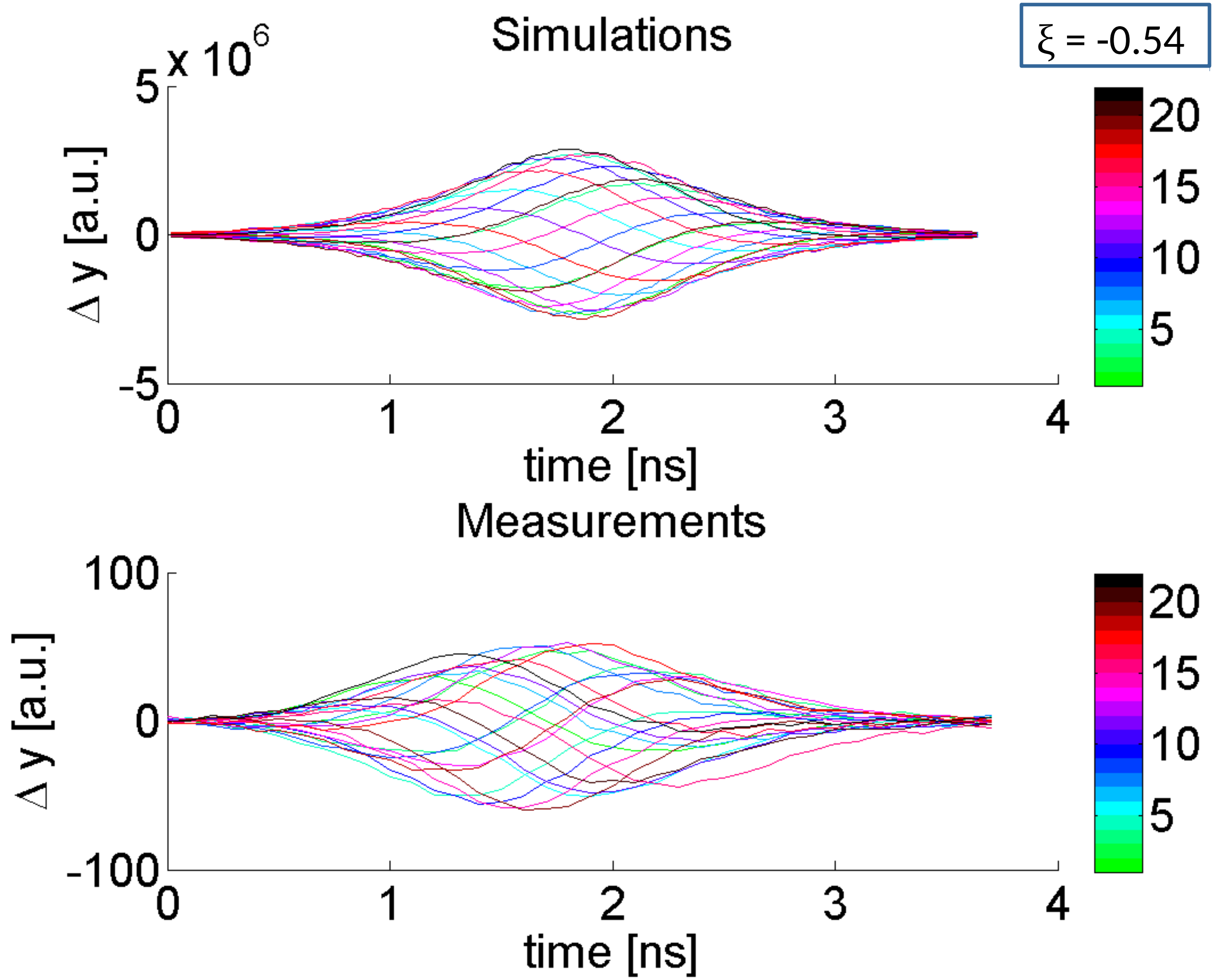


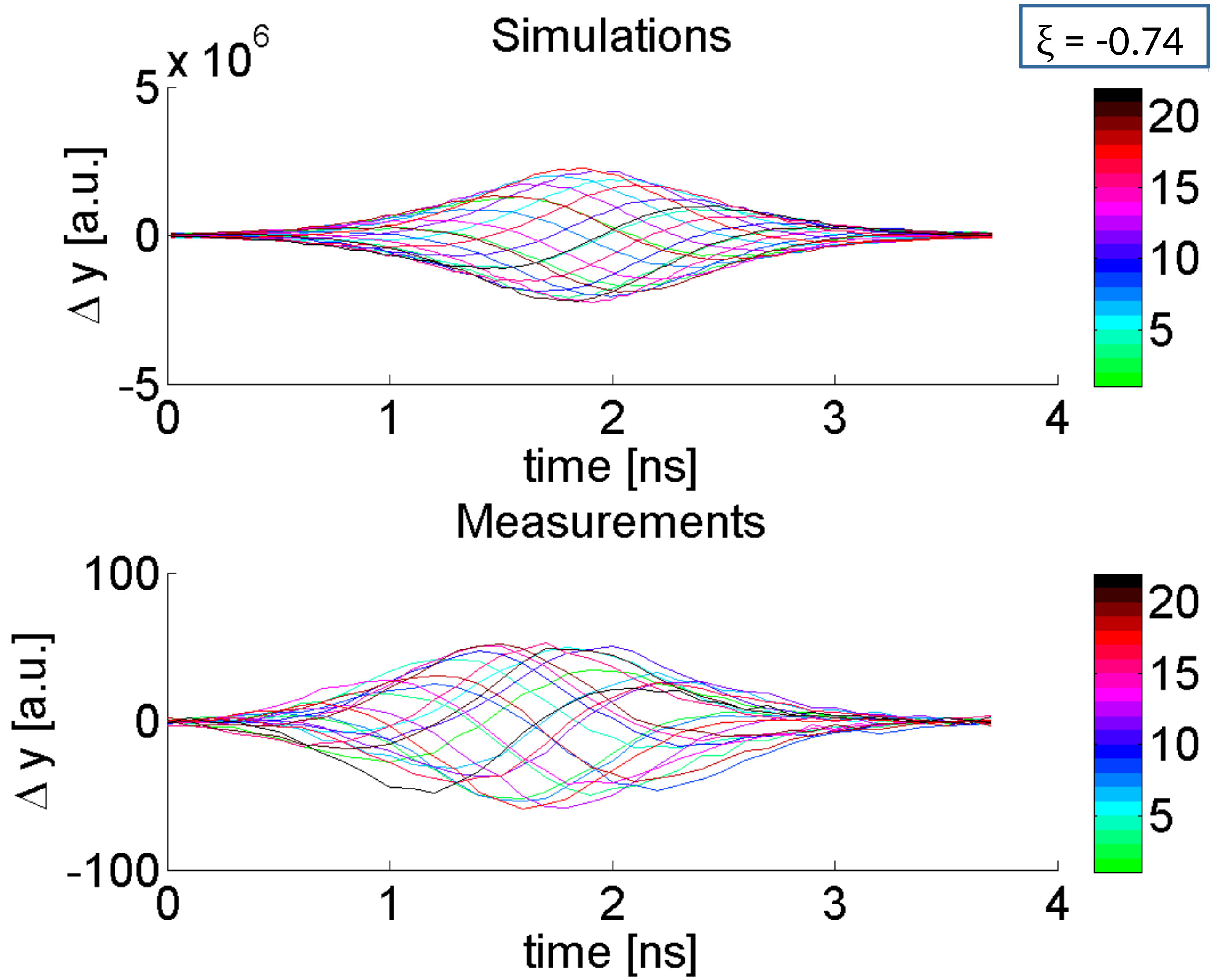
Impedance model: intra-bunch motion



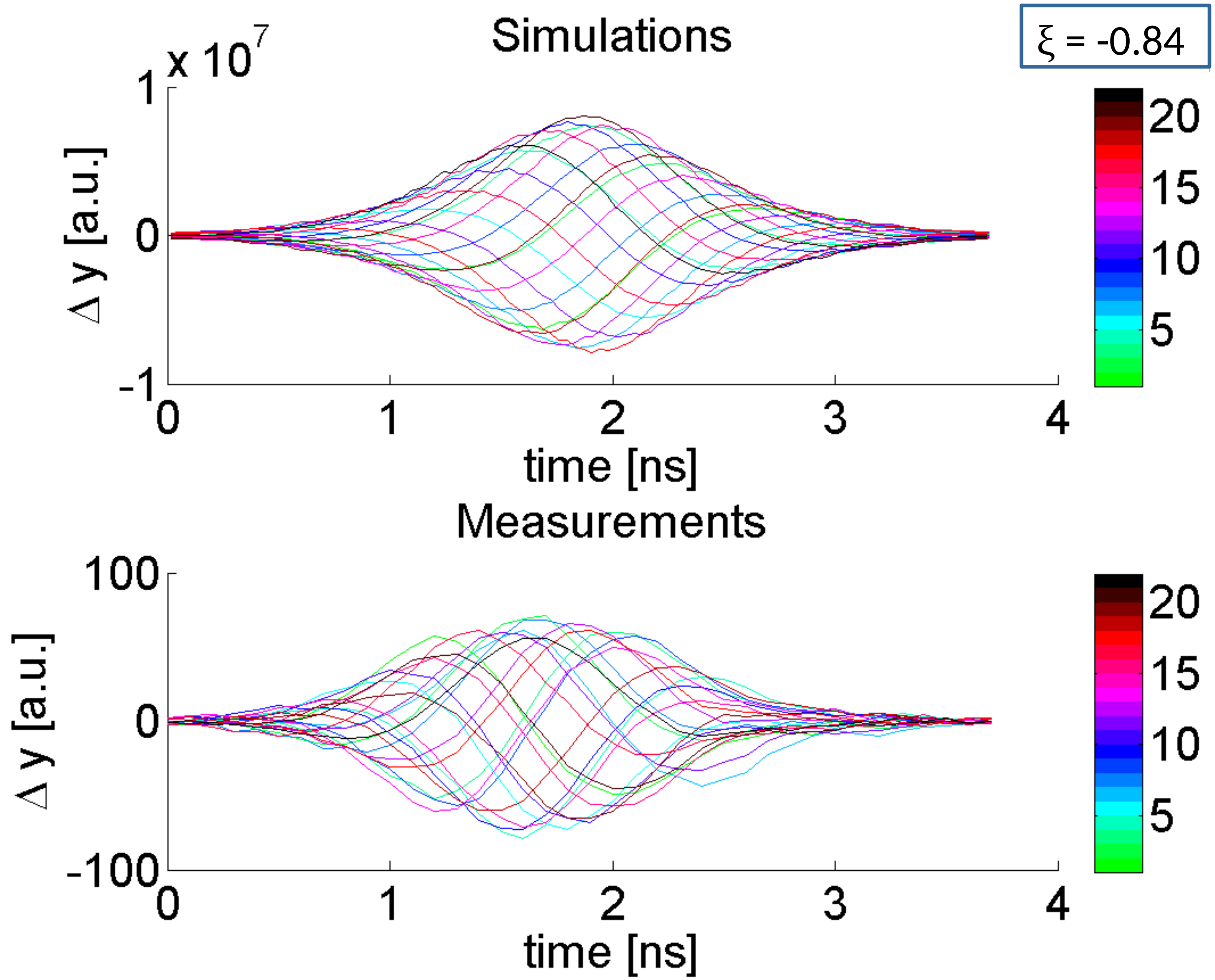
Impedance model: intra-bunch motion



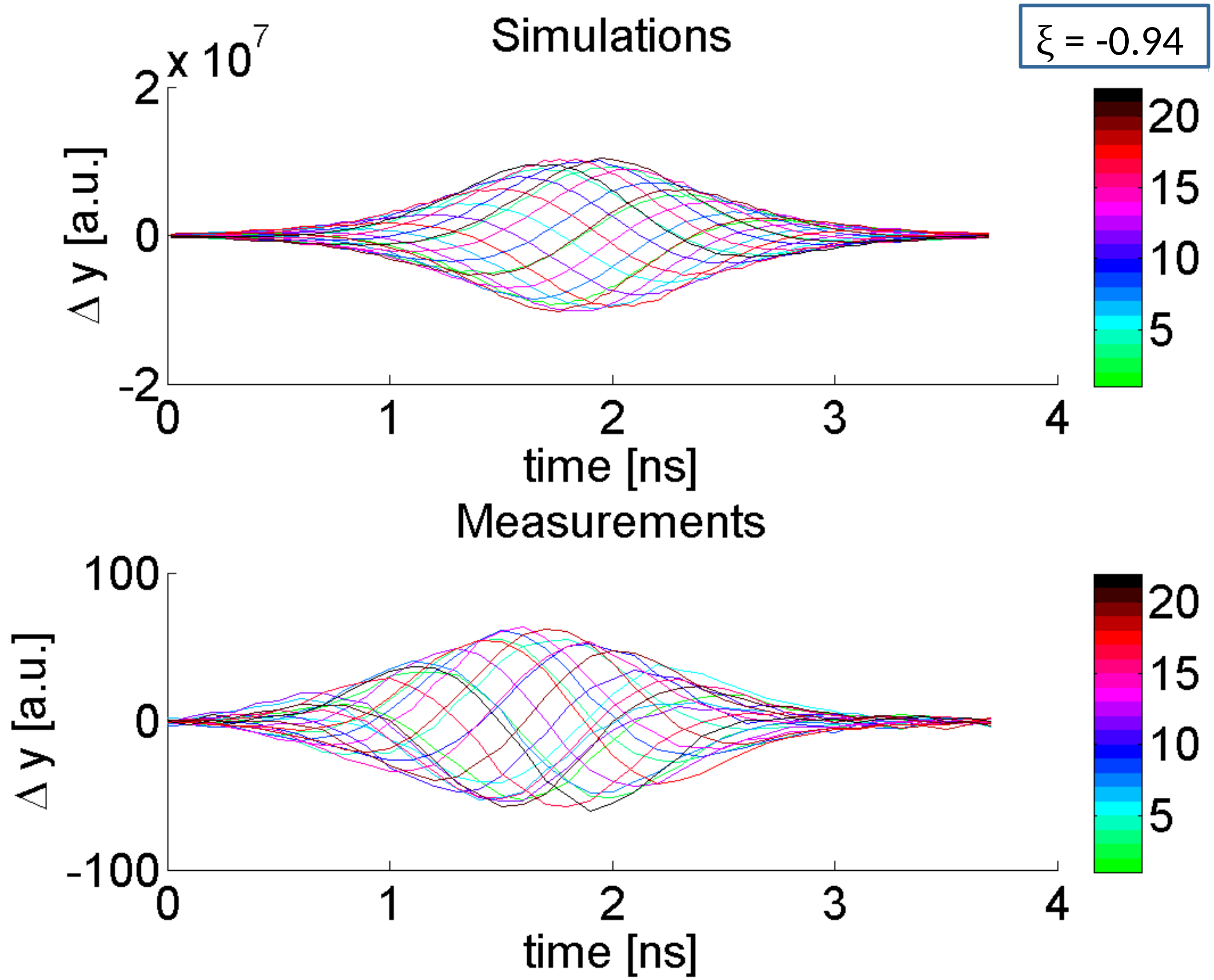
Impedance model: intra-bunch motion



Impedance model: intra-bunch motion



Impedance model: intra-bunch motion

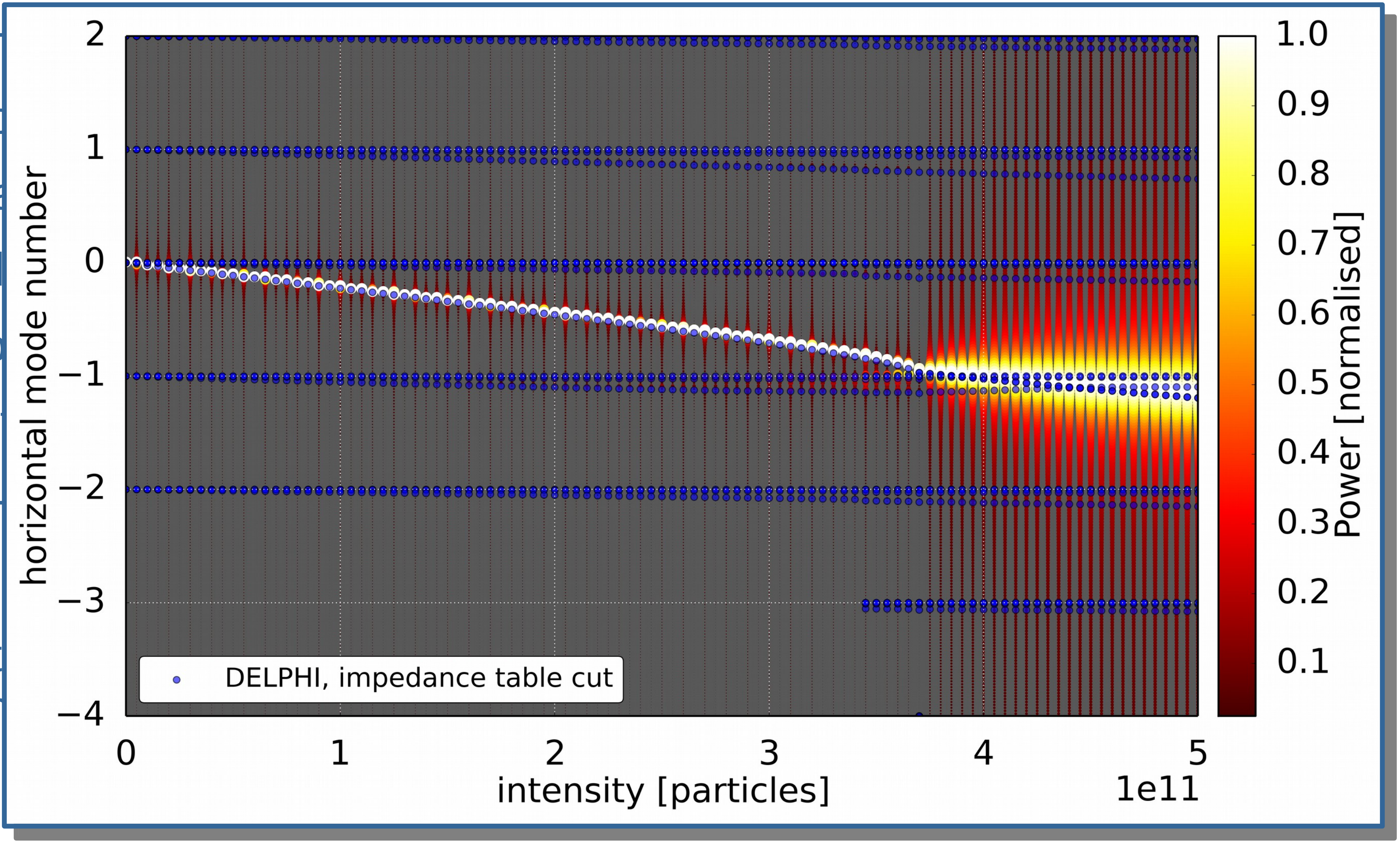


Simulation results – TMCI

- Full LHC impedance model
  - Vacuum pipes
  - Beam screens
  - Collimators
  - Broadband model
- Intensity scan with:
  - PyHEADTAIL (time domain)
  - DELPHI (frequency domain)
- Excitation of several modes – coupling

Simulation results – TMCI

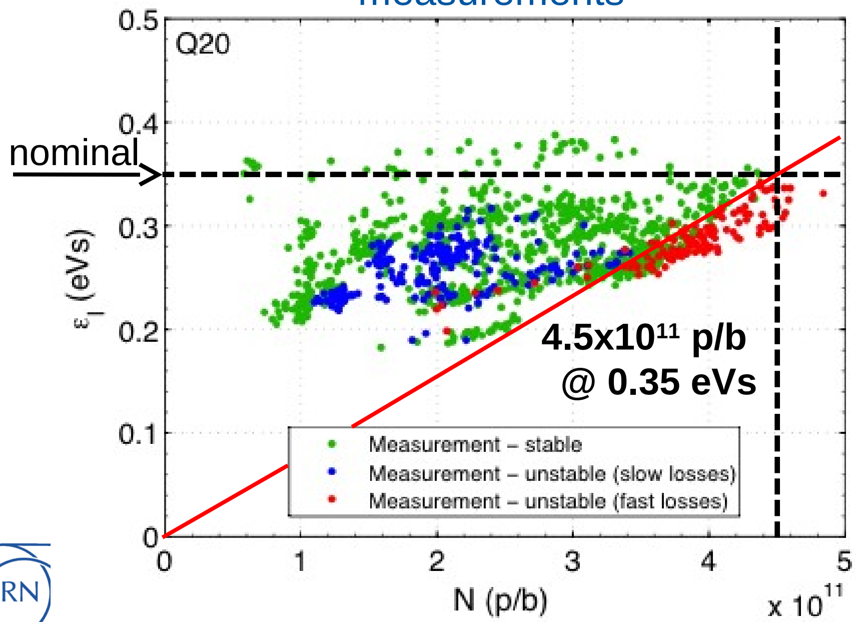
- Full LH
  - Vac
  - Bea
  - Col
  - Bro
- Intensi
  - PyH
  - DE
- Excitat
  couplin



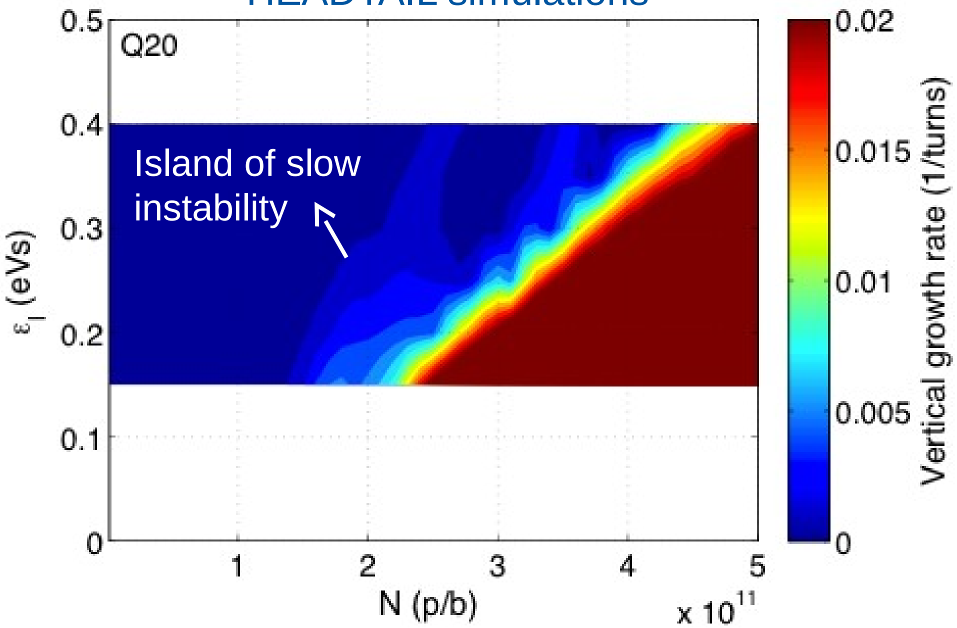
Benchmark of the SPS transverse impedance model: TMCI thresholds

- Two regimes of instability in measurements
- Fast instability threshold with linear dependence on εl
- Slow instability for intermediate intensity and low εl
- Very well reproduced with HEADTAIL simulations
- SPS impedance model includes kickers, wall, BPMs and RF cavities
- Direct space charge not included

measurements

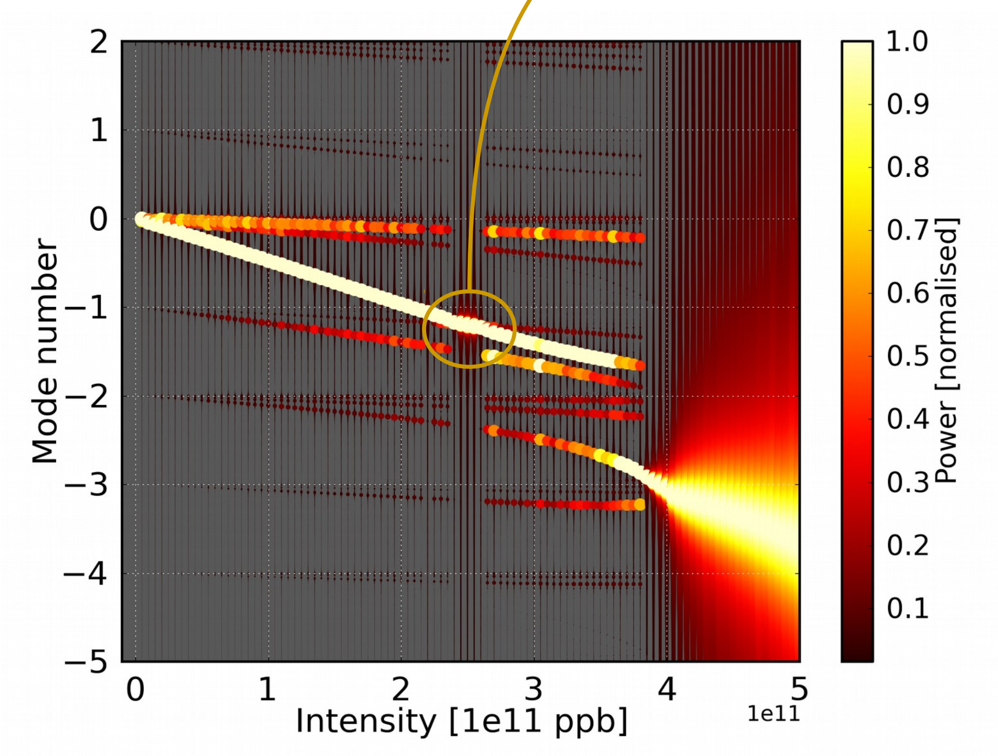


HEADTAIL simulations

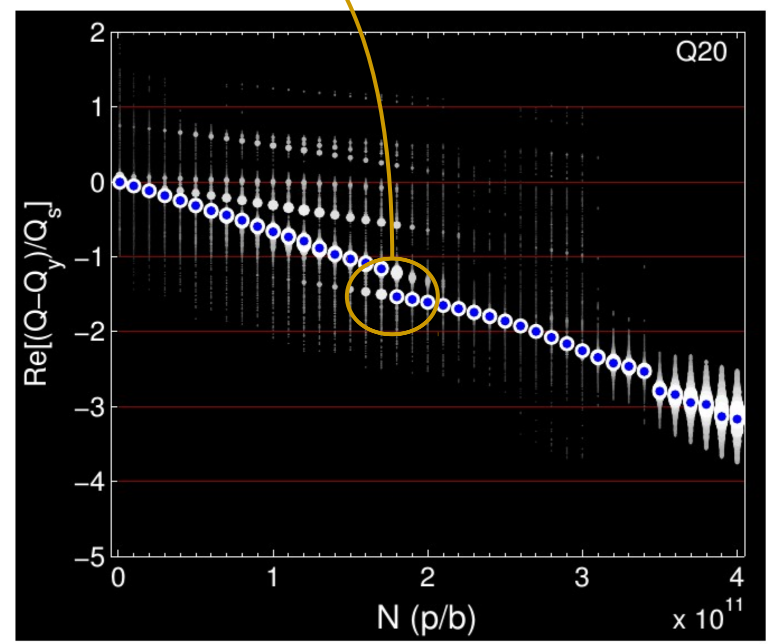


Benchmark of the SPS transverse impedance model: TMCI thresholds

Slow instability



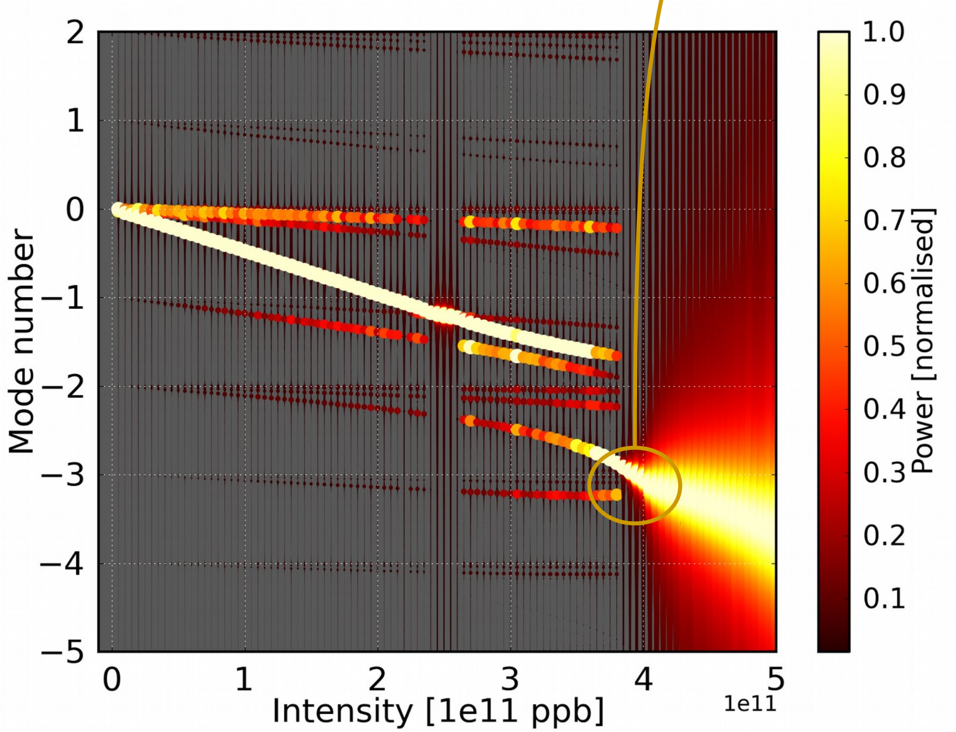
Broadband resonator model



SPS impedance model

Benchmark of the SPS transverse impedance model: TMCI thresholds

Slow instability



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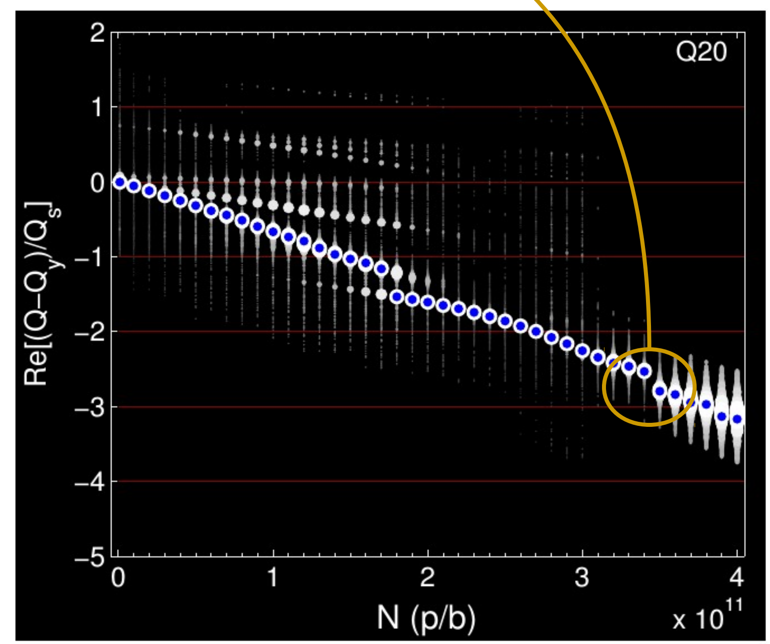
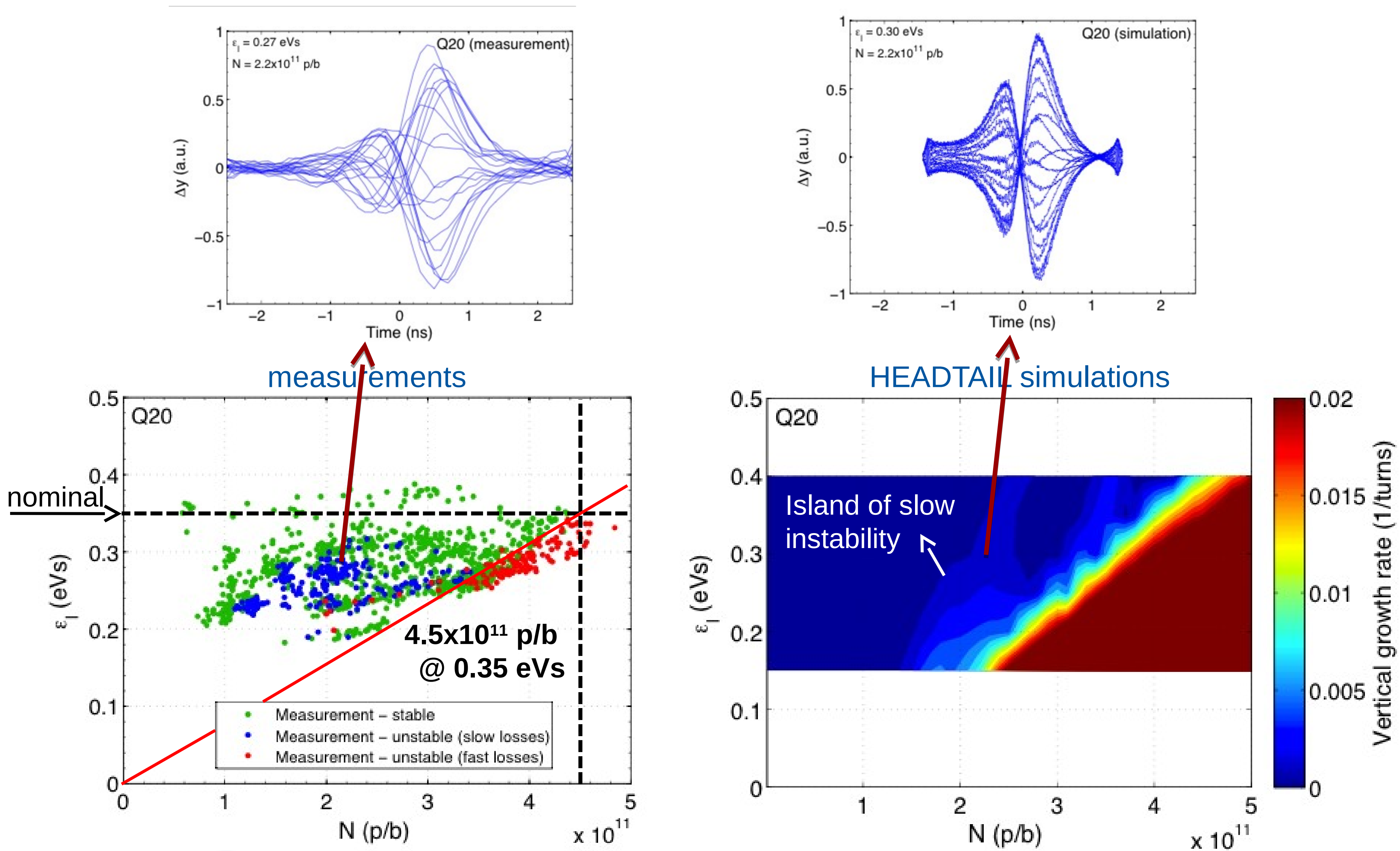
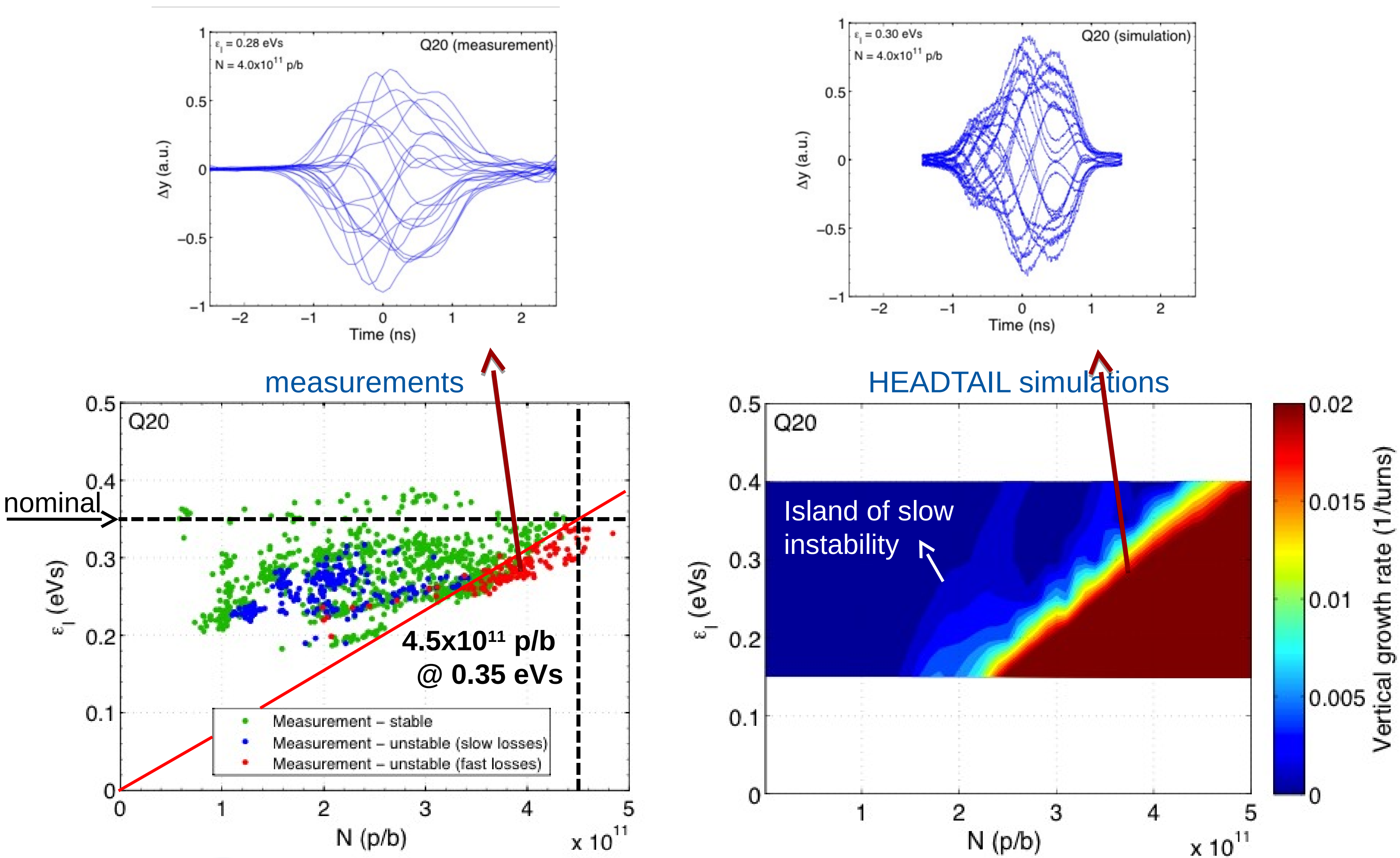


SPS impedance model

Benchmark of the SPS transverse impedance model: TMCI thresholds

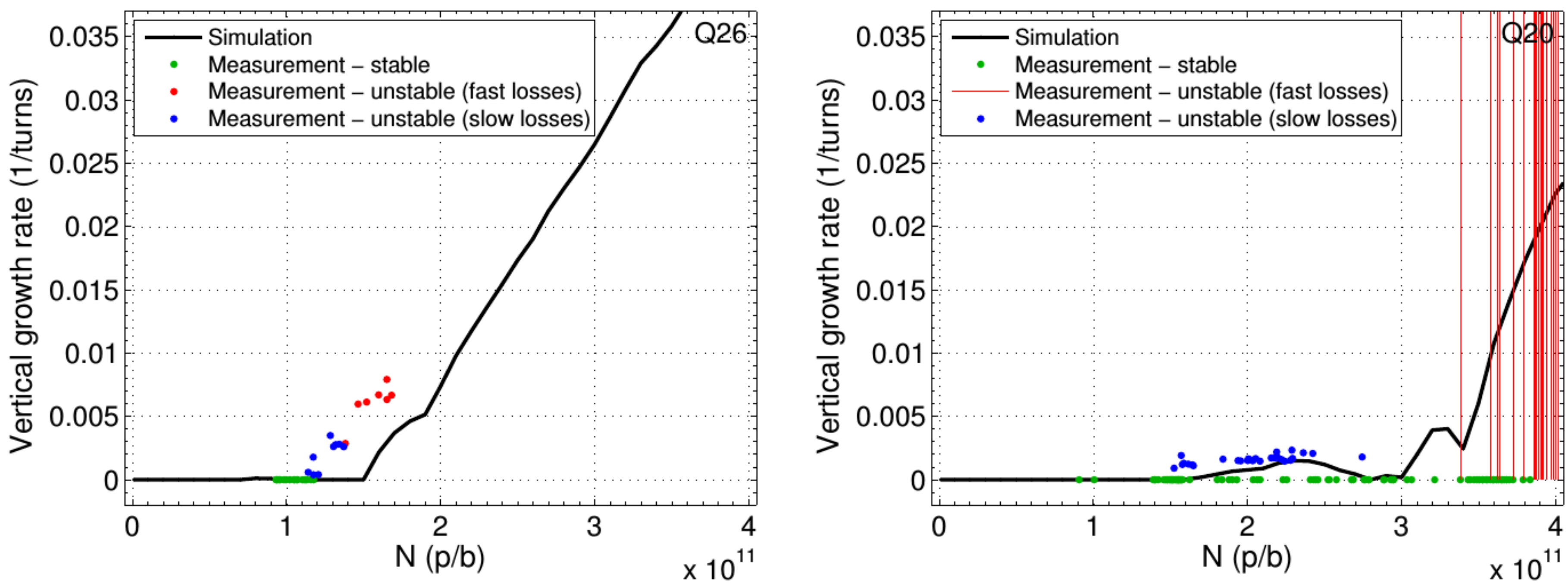


Benchmark of the SPS transverse impedance model: TMCI thresholds



TMCI threshold vs. Qs

- By changing the optics to reduce the transition energy, we can increase the synchrotron tune and by this significantly the instability limit threshold.
- This has been deployed in the SPS where the slippage factor was raised from (Q26) to (Q20) increasing the instability thereshold by a factor



Effect of incoherent tune spread

Finally, we will study the effect of an incoherent tune spread on the beam stability. Starting from the general Vlasov equation in two dimensions, we evaluated the Poisson brackets using

- the coordinate transformations (action-angle variables and polar coordinates)
- the decompositions

$$g_1(J_y, \theta) = g(J_y) e^{i\theta}$$
$$f_1(t, \phi) = e^{-iQ'_y z/(\eta R)} \sum_l a_l R_l(r) e^{il\phi}$$

However, the tune now acquires a term that takes into account the detuning with amplitude:

$$Q = Q_{y0} + Q'_y \delta + \alpha_{yy} J_y + \alpha_{xy} J_x \, .$$

Vlasov eq. – 2-dimensional system

The Vlasov equation with the modified tune simply becomes

$$\sum_{l''} a_{l''} R_{l''}(r) e^{il''\phi} \frac{g(J_y)(\Omega - \omega_{y0} - \omega_0 \alpha_{yy} J_y - \omega_0 \alpha_{xy} J_x - l''\omega_s)}{g_0'(J_y)\sqrt{\frac{2J_y R}{Q_y}}} e^{-i\Omega s/(\beta c)}$$

constant in J_y, therefore $\dfrac{g(J_y)(\Omega - \omega_{y0} - \alpha_{yy} J_y - \alpha_{xy} J_x - l''\omega_s)}{g_0'(J_y)\sqrt{\frac{2J_y R}{Q_y}}} = D$

$$= \frac{e^2 c}{2\beta E C} f_0 \, e^{iQ_y' z/(\eta R)} \times \int dz' \sum_{k=-\infty}^{\infty} \rho^{(1)}(z') e^{-i\Omega(s/(\beta c) - kT_0)} W_1(z - z' - kcT_0)$$

$$\rho^{(1)}(z) = \int dJ_y \, d\theta \, g(J_y) e^{i\theta} \, y \, \rho(z) = D \frac{R}{Q_{y0}} \frac{\omega_0}{I(Q_c)} \rho(z)$$

$$I(Q_c) = \int dJ_y \frac{g_0'(J_y) \, J_y}{Q_c - Q_{y0} - \alpha_{yy} J_y - \alpha_{xy} J_x - l'' Q_s}$$

Vlasov eq. – 2-dimensional system

The Vlasov equation with the modified tune simply becomes

$$\sum_{l''} a_{l''} R_{l''}(r) e^{il''\phi} \boxed{\frac{g(J_y)(\Omega - \omega_{y0} - \omega_0 \alpha_{yy} J_y - \omega_0 \alpha_{xy} J_x - l''\omega_s)}{g_0'(J_y)\sqrt{\frac{2J_y R}{Q_y}}}} e^{-i\Omega s/(\beta c)}$$

cancels with D_l on RHS

What has changed?

$$= \frac{e^2 c}{2\beta E C} f_0 \, e^{iQ_y' z/(\eta R)} \times \int dz' \sum_{k=-\infty}^{\infty} \underbrace{\rho^{(1)}(z')}_{} e^{-i\Omega(s/(\beta c) - kT_0)} W_1(z - z' - kcT_0)$$

$$I(Q_c) = \int dJ_y \frac{g_0'(J_y)\, J_y}{\boxed{Q_c - Q_{y0} - \alpha_{yy} J_y - \alpha_{xy} J_x - l'' Q_s}}$$

What was originally the eigenvalue, resulting in a linear EV problem, gets now drawn into the integral.

Vlasov eq. – 2-dimensional system

We move from the original EV problem (solving for Λ)

$$\det\Big(\underbrace{(\Omega - \omega_{y0} - l\omega_s)}_{\Lambda}\delta_{kk'}\delta_{ll'} - \mathcal{M}_{kk',ll'}\Big) = 0$$

to the non-linear equation which we can solve numerically to obtain the full correct solution.

$$\det\left(\left(\frac{\omega_0}{I\left(\frac{\Omega}{\omega_0}\right)}\right)\delta_{kk'}\delta_{ll'} - \mathcal{M}_{kk',ll'}\right) = 0$$

Vlasov eq. – 2-dimensional system

Alternatively, we find the eigenvalues Λ for the case without tune spread.

$$\det\left(\Lambda\,\delta_{kk'}\delta_{ll'} - \mathcal{M}_{kk',ll'}\right) = 0$$

We then identify:

$$\Lambda(Q_c) \equiv \frac{\omega_0}{I\left(\overline{Q}_c\right)}\,,$$

i.e. we say that the effect of adding a tune spread does not modify the coherent mode structure itself (the eigenvectors remain unchanged) but only the complex tune shifts (the eigenvalues change). Thus, the original complex tune shift without tune spread Q_c translates to the new complex tune shift $\overline{Q}_c$. A negative imaginary part will lead to a stable beam whereas a positive imaginary part will render the beam unstable. A purely real $\overline{Q}_c$ will trace out the stability boundary. We can search for all Q_c at the stability boundary by inserting purely real $\overline{Q}_c$ and evaluating the dispersion integral. The resulting curve will devide the plane in two regoins where a given mode Q_c will either be stabilised to $\overline{Q}_c$ or not. A more detailed discussion will lead to the theory of stability diagrams and Landau damping. The treatment of these topics is beyond the scope of this course. Hopefully, however, the study of these topics has well been motivated by now.

THE END